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NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-188

Diffraction at High Frequencies by a Circular Disc

D. S. JONES

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OFFICE OF AEROSPACE RESEARCH
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Abstract

It is shown how to convert the integral equation for the pressure distribution on a circular disc in an axisymmetric sound wave into a singular integral equation of the first kind. This singular integral equation has a simple kernel and can be transformed easily into an integral equation of the second kind which is especially useful at high frequencies. Although a direct iteration on this last integral equation fails an indirect method is devised which ensures that each iterate is of lower order than the preceding at high frequencies. The form of the general term in the iteration is given together with the first terms of its asymptotic behaviour. It is relatively simple to estimate the error caused by stopping at any particular iterate .

Detailed calculations are made for a plane wave striking the disc at normal incidence.

Introduction

The problem of the diffraction by a circular disc has been investigated intensely during the past decade, primarily because it is the simplest problem in which the diffracting edge is of finite length and therefore serves as a convenient test of methods which may be applicable to more general shapes. In principle, any numerical information required can be obtained from the solution in terms of spheroidal functions and indeed a considerable amount of such information was derived by Bouwkamp [1] many years ago. In recent years the emphasis has been on attempts to provide analytical information which can be used to supply backing for approximate theories of more general shapes. Apart from one method employed by Chako [2] and by Nomura and Katsura [3] in which an aperture distribution in terms of hypergeometric polynomials is assumed the main approach has been via integral equations. The author (Jones [4]) discovered a relatively simple integral equation which is especially suitable for finding results quickly at low frequencies. Other derivations of this integral equation have been given by N.I. and A.N. Akhiezer [5], MacCamy and Heins [6], Bazer and Brown [7], Noble [8], Collins [9], Bazer and Hochstadt [10], Williams [11]. A different integral equation, also suitable for low frequencies though less simple than the preceding one, was found by Magnus [12]; it has been derived by other methods by Noble (loc. cit.), Lebedev and Skal'skaya [13] and has recently been generalised to some other obstacles by Collins [14].

At high frequencies the standard integral equation has been solved approximately by modifying the kernel so that it is amenable to the Wiener-Hopf technique (Levine [15], Levine and Wu [16]). By this method Levine and Wu obtained the scattering coefficient as far as $O(\alpha^{-5/2})$, where α is the product of the wave number and radius of the disc. Their method was used subsequently by Seshradi and Wu [17] for the diffraction of electromagnetic waves.

Approximate methods based on physical reasoning have been suggested by several authors. One method (Clemmow [18], Millar [19, 20]) uses assumed currents on the screen. A theory which employs rays as the basic concept has been given by Keller [21, 22], Keller, Lewis and Seckler [23], Karp and Keller [24].

Recently a very general method of tackling the asymptotic solution of diffraction problems has been proposed by Keller and Buchal [25]. Their method is applicable, in principle, to discs of arbitrary shape and it is therefore important to confirm that it gives the correct answer for the circular disc. Certainly, it supplies a scattering coefficient which is in agreement with the calculations of Levine and Wu as far as they go. For confirmation of higher order terms it would be necessary to extend the analysis of Levine and Wu. However, this extension would not be simple since their method involves two approximations - the first replaces the kernel by a more convenient one and the second consists in converting the resulting integral equation into a series of Wiener-Hopf integral equations by an iterative procedure.

It seemed, therefore, that a new approach might well be profitable. The object of the present analysis is to provide a new integral equation which is simple and especially suitable for handling high frequencies. The general term in the asymptotic development can be obtained and, in principle, the field can be evaluated to any desired order, the main labour involved being the asymptotic estimation of certain integrals.

In section 1 is derived a singular integral equation of the first kind for the sound-soft disc. This singular integral equation is converted to an integral equation of the second kind in section 2 and the particular form that it takes for an incident plane wave is determined in section 3. Section 4 is concerned with an iterative procedure for solving the integral equation of the second kind; this procedure is especially appropriate at high frequencies. The resulting distribution on the disc is found in section 5 and the high frequency scattering coefficient in section 6.

In section 7 the corresponding integral equations for the sound-hard disc are derived and section 8 is devoted to the appropriate iterative procedure. The distribution on the disc and scattering coefficient are determined in sections 9 and 10 respectively.

It is shown in section 11 that the solutions of the integral equations of the second kind described above are unique and that the iterative procedure does provide an asymptotic development at high frequencies.

The derivation of certain results has been carried out in appendices. References to equations in appendices are written, for example, (B1), meaning equation (1) of Appendix B.

1. Derivation of the Integral Equation

We consider the problem of the diffraction of a small-amplitude harmonic sound wave by a sound-soft circular disc of radius a. The boundary condition is that the field vanishes on the disc. The case of the sound-hard or rigid disc where the normal derivative of the field vanishes on the disc will be treated in later sections.

Let r, φ and z be cylindrical polar coordinates, and normalize all lengths by division by a so that the disc occupies $0 \leq r \leq 1$, $z = 0$. Let the incident field be independent of φ and be represented by $u_o(r,z)$ at (r,φ,z), the time dependence $e^{i\omega t}$ being understood and omitted throughout. The total field $u(\underline{R})$, at the point whose position vector with respect to the orgin is $\underline{R}$, is then independent of φ and may also be written $u(r,z)$.

The field satisfies the equation

$$(\nabla^2 + \alpha^2)u = 0$$

where α is the product of the wave number k and the radius a. Throughout α will be taken as a positive real number. The total field at the point $\underline{R}$ can always be written in the form

$$u(\underline{R}) = u_o(r,z) - \frac{1}{4\pi}\int_S \left[\frac{\partial u}{\partial z_1}\right] \psi(\underline{R},\underline{R}_1)dS$$

since $u = 0$ on the disc. Here S is the unit circle on $z = 0$, $\underline{R}_1$ is a point of S, $\psi(\underline{R},\underline{R}_1) = e^{-i\alpha|\underline{R} - \underline{R}_1|}/|\underline{R} - \underline{R}_1|$ and

$$\left[\frac{\partial u}{\partial z_1}\right] = \left[\frac{\partial u(r_1,z_1)}{\partial z_1}\right]_{z_1=+0} - \left[\frac{\partial u(r_1,z_1)}{\partial z_1}\right]_{z_1=-0}$$

Applying the boundary condition u = 0 we obtain as an integral equation to determine $f(r_1)$ (= $[\partial u/\partial z_1]$)

$$(1) \qquad u_o(r,0) - \frac{1}{4\pi}\int_S f(r_1)\,\psi(\underline{R},\underline{R}_1)dS = 0$$

where now $\underline{R}$ is a point of S. As a consequence

$$(2) \quad \psi(\underline{R},\underline{R}_1) = \frac{\exp[-i\alpha\,\{r^2 + r_1^2 - 2rr_1\cos(\varphi-\varphi_1)\}^{1/2}]}{\{r^2 + r_1^2 - 2rr_1\cos(\varphi-\varphi_1)\}^{1/2}}$$

When ψ is given by (2) it has been shown (Jones [4]) that

$$(3) \quad \begin{aligned} \int_o^{2\pi}\psi d\varphi_1 &= 4\int_o^{\min(r,r_1)} \frac{\cos\alpha(r^2-x^2)^{1/2}\,\cos\alpha(r_1^2-x^2)^{1/2}}{(r^2-x^2)^{1/2}\,(r_1^2-x^2)^{1/2}}\,dx \\ &\quad -4i\int_o^r \frac{\cos\alpha(r^2-x^2)^{1/2}\sin\alpha(r_1^2-x^2)^{1/2}}{(r^2-x^2)^{1/2}\,(r_1 \;\, - x^2)^{1/2}}\,dx. \end{aligned}$$

For convenience a derivation is given in Appendix A.

When (3) is substituted in (1) the result is

$$u_o(r,0) - \frac{1}{\pi}\int_o^r \frac{\cos\alpha(r^2-x^2)^{1/2}}{(r^2-x^2)^{1/2}}\left\{\int_x^1 \frac{f(r_1)r_1\cos\alpha(r_1^2-x^2)^{1/2}}{(r_1^2-x^2)^{1/2}}\,dr_1 - i\int_o^1 \frac{f(r_1)r_1\sin\alpha(r_1^2-x^2)^{1/2}}{(r_1^2-x^2)^{1/2}}\,dr_1\right\}dx = 0$$

for $r \leq 1$. The change of variables $r^2 = y$, $x^2 = z$, $r_1^2 = t$ converts this to

$$(4) \quad u_o(y^{1/2},0) - \frac{1}{4\pi}\int_o^y \frac{\cos\alpha(y-z)^{1/2}}{(y-z)^{1/2}}\left\{\int_z^1 \frac{f(t^{1/2})\cos\alpha(t-z)^{1/2}}{(t-z)^{1/2}}\,dt - i\int_o^1 \frac{f(t^{1/2})\sin\alpha(t-z)^{1/2}}{(t-z)^{1/2}}\,dt\right\}\frac{dz}{z^{1/2}} = 0$$

for $y \leq 1$. The solution of an integral equation of the form

$$(5) \qquad f_1(y) = \int_0^y h_1(z) \frac{\cos \alpha(y-z)^{1/2}}{(y-z)^{1/2}} dz$$

is known (Jones [4]) to be

$$h_1(z) = \frac{1}{\pi} \frac{d}{dz} \int_0^z f_1(y) \frac{\cosh \alpha(z-y)^{1/2}}{(z-y)^{1/2}} dy.$$

Hence the solution of (4) is

$$(6) \qquad \int_z^1 \frac{f(t^{1/2}) \cos \alpha(t-z)^{1/2}}{(t-z)^{1/2}} dt - i \int_0^1 \frac{f(t^{1/2}) \sin \alpha(t-z)^{1/2}}{(t-z)^{1/2}} dt$$

$$= 4z^{1/2} \frac{d}{dz} \int_0^z u_o(y^{1/2}, 0) \frac{\cosh \alpha(z-y)^{1/2}}{(z-y)^{1/2}} dy$$

valid for $0 \leq z \leq 1$.

The integral equation previously described in the introduction as being especially useful at low frequencies is derived from (6) by applying the inverse operator of the first term of the left hand side. The inverse operator is easily found by changing t and z to $1 - t$ and $1 - z$ so as to obtain an integral equation of the type in (5). To find an integral equation suitable for high frequencies (large α) rewrite (6) as

$$(7)\int_0^1 \frac{f(t^{1/2})e^{-i\alpha(t-z)^{1/2}}}{(t-z)^{1/2}}\,dt - \int_0^z f(t^{1/2})\,\frac{\cos\alpha(t-z)^{1/2}}{(t-z)^{1/2}}\,dt$$

$$= 4z^{1/2}\,\frac{d}{dz}\int_0^z u_o(y^{1/2}, 0)\,\frac{\cosh\alpha(z-y)^{1/2}}{(z-y)^{1/2}}\,dy.$$

For definiteness $(t-z)^{1/2}$ means $i(z-t)^{1/2}$ when $z > t$. It is proved in Appendix B that, for the x and t involved,

$$(8)\quad \int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}\,e^{-i\alpha(t-z)^{1/2}}}{(x-z)^{1/2}\,(t-z)^{1/2}}\,dz = \int_{t^{1/2}-\,x^{1/2}}^{t^{1/2}+\,x^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy - \pi i H(x-t)$$

where H(x) is the Heaviside unit function and the principal value of the integral is to be taken. In fact, we shall always use the convention that a principal value is implied when the integrand contains a simple pole on the interval of integration. Another relation proved in Appendix B is

$$(9)\int_t^x \frac{\cos\{\alpha(x-z)^{1/2}\}\,\cos\{\alpha(t-z)^{1/2}\}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz = -\pi i\ \mathrm{sgn}(x-t).$$

Multiply both sides of (7) by $\cos\{\alpha(x-z)^{1/2}\}\ /(x-z)^{1/2}$ and integrate with respect to z from 0 to x. On account of (8) and (9) equation (7) becomes

(10)
$$\int_0^1 f(t^{1/2})\left\{\int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy - \pi i H(x-t)\right\} dt$$

$$+\int_0^x f(t^{1/2})\,\pi i\,\mathrm{sgn}(x-t)\,dt$$

$$= 4\int_0^x \frac{z^{1/2}\cos\{\alpha(x-z)^{1/2}\}}{(x-z)^{1/2}}\,\frac{d}{dz}\int_0^z u_o(y^{1/2}, 0)\,\frac{\cosh\alpha(z-y)^{1/2}}{(z-y)^{1/2}}\,dy\,dz.$$

Since the left hand side of (10) is

$$\int_0^1 f(t^{1/2})\int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy\,dt$$

a derivative of (10) with respect to x supplies

$$\int_0^1 \frac{f(t^{1/2})}{2x^{1/2}}\left\{\frac{e^{-i\alpha(t^{1/2}+x^{1/2})}}{t^{1/2}+x^{1/2}} + \frac{e^{-i\alpha(t^{1/2}-x^{1/2})}}{t^{1/2}-x^{1/2}}\right\} dt$$

$$= 4\frac{d}{dx}\int_0^x \frac{z^{1/2}\cos\{\alpha(x-z)^{1/2}\}}{(x-z)^{1/2}}\,\frac{d}{dz}\int_0^z u_o(y^{1/2}, 0)\,\frac{\cosh\alpha(z-y)^{1/2}}{(z-y)^{1/2}}\,dy\,dz$$

where, as already stated, a principal value is implied if necessary.

The change of variables $x = v^2$ and $t = w^2$ now leads to

(11)
$$\int_0^1 wf(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw = F^{(o)}(v)$$

where

(12)
$$F^{(o)}(v) = 2\frac{d}{dv}\int_0^{v^2} \frac{z^{1/2}\cos\{\alpha(v^2-z)^{1/2}\}}{(v^2-z)^{1/2}}\,\frac{d}{dz}\int_0^z u_o(y^{1/2},0)\,\frac{\cosh\alpha(z-y)^{1/2}}{(z-y)^{1/2}}\,dy\,dz.$$

The integral equation (11) holds for $0 \leq v \leq 1$.

It is easy to see from the structure of (11) why it may be expected to be especially helpful at high frequencies. For large α the dominant contribution will come from the neighbourhood of the principal value, corresponding to the physical situation that to a first approximation the distribution on the disc is determined locally by the incident field. Furthermore the most significant values will tend to occur near an edge $v \simeq 1$ since there the neighbourhood of the principal value is about to move out of the interval of integration. (The w in the numerator prevents $v = 0$ being similarly important). The first term of the integrand represents the effect of secondary radiation and shows, in particular, that for a point near the edge the main secondary waves come from the diametrically opposite point. However, it should be noticed that as v approaches 0 both terms in (11) are, roughly speaking, of the same order of significance. In essence this corresponds to the fact that every point of the edge is equally important with regard to producing a field at the centre of the disc.

In the low frequency case when α is small both terms on the left-hand side of (11) are of equal importance. Some consideration of the low frequency problem will be found in Appendix C.

2. Conversion to an Integral Equation of the Second Kind

There are several ways in which (11) can be transformed into an integral equation of the second kind. In view of the remarks made at the end of the preceding section concerning the relative significance of the two parts of the kernel at high frequencies we put

$$(13) \qquad \int_0^1 \frac{wf(w)e^{-i\alpha(w-v)}}{w-v}\,dw = F(v).$$

Expressed in the form

$$\int_0^1 \frac{wf(w)e^{-i\alpha w}}{w-v}\,dw = F(v)e^{-i\alpha v}$$

(13) is a singular integral equation of well-known type (see, for example, Mikhlin [26]) with a general solution

$$wf(w)e^{-i\alpha w} = -\frac{1}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \frac{v^{1/2}(1-v)^{1/2}}{v-w} F(v)e^{-i\alpha v}\,dv + \frac{C}{w^{1/2}(1-w)^{1/2}}$$

where C is a constant. C is determined by applying a condition at either $w = 0$ or $w = 1$. Since no edge condition is necessary to ensure uniqueness (Jones [27]) for the sound-soft disc we impose the condition that $f(w)$, and therefore $wf(w)$, is bounded at $w = 0$. This leads to

$$(14) \qquad f(w) = -\frac{e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \frac{(1-v)^{1/2}}{v^{1/2}} \frac{F(v)e^{-i\alpha v}}{v-w}\,dv.$$

By substituting for $f(w)$ from (14) we find

$$\int_0^1 wf(w)\frac{e^{-i\alpha(w+v)}}{w+v}\,dw = -\frac{1}{\pi^2}\int_0^1 \frac{e^{-i\alpha v}}{w+v} \frac{w^{1/2}}{(1-w)^{1/2}} \int_0^1 \frac{(1-t)^{1/2}}{t^{1/2}} \frac{F(t)e^{-i\alpha t}}{t-w}\,dt\,dw$$

$$= -\frac{e^{-i\alpha v}}{\pi^2}\int_0^1 \frac{(1-t)^{1/2}}{t^{1/2}} F(t)e^{-i\alpha t}\int_0^1 \frac{w^{1/2}}{(1-w)^{1/2}} \frac{dw}{(w+v)(t-w)}\, dt$$

on inverting the order of integration. The inner integral may be easily evaluated by introducing the variable x defined by $w = x^2/(1+x^2)$. We thus find, remembering that a principal value is involved,

$$(15)\quad \int_0^1 \frac{w^{1/2}}{(1-w)^{1/2}} \frac{dw}{(w+v)(w-t)} = \frac{\pi}{v+t} \frac{v^{1/2}}{(1+v)^{1/2}} \quad (v > 0, 1 > t \geq 0).$$

(An alternative derivation will be found in Appendix E; see (E 14))

Hence

$$(16)\quad \int_0^1 wf(w) \frac{e^{-i\alpha(w+v)}}{w+v} dw = \frac{e^{-i\alpha v}}{\pi} \frac{v^{1/2}}{(1+v)^{1/2}} \times \\ \times \int_0^1 \frac{(1-t)^{1/2} F(t)e^{-i\alpha t}}{t^{1/2}(t+v)} dt.$$

Since, from (11) and (13),

$$(17)\quad F(v) = F^{(o)}(v) - \int_0^1 wf(w) \frac{e^{-i\alpha(w+v)}}{w+v} dw$$

we obtain as an integral equation for F

$$(18)\quad F(v) = F^{(o)}(v) - \frac{v^{1/2}e^{-i\alpha v}}{\pi(1+v)^{1/2}} \int_0^1 \frac{(1-t)^{1/2}}{t^{1/2}(t+v)} F(t)e^{-i\alpha t}\, dt \quad (0 \leq v \leq 1).$$

When (18) has been solved for F the function f, and hence the whole field, is known from (14).

Equation (18) differs from (11) in being an integral equation of the second kind. Its kernel is perhaps not quite so simple looking but it is no longer singular and, as we shall see later, the appearance of the kernel can be somewhat simplified by taking $(1+v)^{1/2}F(v)/v^{1/2}\, e^{-i\alpha v}$ as a new unknown.

It is also possible to form an integral equation of the second kind for f by substituting for F from (17) into (14). The result is

$$f(w) = - \frac{e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{e^{-i\alpha v}}{v-w} \left\{F^{(o)}(v) - \int_0^1 tf(t)\, \frac{e^{-i\alpha(t+v)}}{t+v}\, dt\right\} dv.$$

However it does not seem possible to simplify the kernel, even to the extent of eliminating one of the integrations. Consequently, the kernel is considerably more complicated than that of (18) and approximation by iteration would involve a double integration for each iterate. In fact, the form of the first term for an incident plane wave is such that there does not seem to be a uniformly valid representation any simpler than the original. It is therefore advantageous to operate with (18).

Both (11) and (18) seem to be new. The integral equation derived by Magnus [12] bears some resemblance to (18) but is in fact more complicated.

3. The Incident Plane Wave

In order to see how a detailed solution of (18) goes consider the special case of an incident plane wave in which $u_o(r,z) = e^{-i\alpha z}$. Then, from (12),

$$F^{(o)}(v) = 2\frac{d}{dv}\int_o^{v^2} \frac{\cos\{\alpha(v^2-z)^{1/2}\}\cosh \alpha z^{1/2}}{(v^2-z)^{1/2}}\,dz$$

$$= 2\frac{d}{dv}\left[-\frac{2}{\alpha}\sin\{\alpha(v^2-z)^{1/2}\}\cosh \alpha z^{1/2}\right]_o^{v^2}$$

$$+ 2\frac{d}{dv}\int_o^{v^2}\sin\{\alpha(v^2-z)^{1/2}\}\frac{\sinh \alpha z^{1/2}}{z^{1/2}}\,dz$$

after an integration by parts. Hence

$$F^{(o)}(v) = 4\cos\alpha v + 2\alpha v\int_o^{v^2}\frac{\cos\{\alpha(v^2-z)^{1/2}\}\sinh \alpha z^{1/2}}{(v^2-z)^{1/2}\,z^{1/2}}\,dz.$$

Since $(\sinh \alpha z^{1/2})/z^{1/2}$ can be regarded as $\left\{\sin\alpha(-z)^{1/2}\right\}/(-z)^{1/2}$ (B3), with $x = v^2$ and $t = 0$, shows that

$$F^{(o)}(v) = 4\cos\alpha v + 2\alpha v\int_{-v}^{v}\frac{\sin\alpha t}{t}\,dt$$

$$= 4\cos\alpha v + 4\alpha v\,\mathrm{Si}(\alpha v)$$

where $\mathrm{Si}(x) = \int_{o}^{x}\frac{\sin t}{t}\,dt$. An alternative way of writing Si will be found valuable in the following. Since $\int_{-v}^{v}\frac{\cos t}{t}\,dt = 0$,

(20) $$\mathrm{Si}(x) = \frac{1}{2}\,i\int_{-x}^{x}\frac{e^{-it}}{t}\,dt = \frac{1}{2}\,i\int_{-2x}^{o}\frac{e^{-i(t+x)}}{t+x}\,dt.$$

Consequently

(21) $$\mathrm{Si}(\alpha v) = \frac{1}{2}\,i\int_{-2v}^{o}\frac{e^{-i\alpha(t+v)}}{t+v}\,dt\,.$$

Now define a new unknown $F_1(v)$ by means of

(22) $$F(v) = 2e^{i\alpha v} + 2i\alpha v\int_{-2v}^{\infty}\frac{e^{-i\alpha(t+v)}}{t+v}\,dt + F_1(v).$$

The equation satisfied by $F_1(v)$ is, from (18), (19) and (21),

(23)
$$F_1(v) = 2e^{-i\alpha v} - 2i\alpha v\int_{o}^{\infty}\frac{e^{-i\alpha(t+v)}}{t+v}\,dt - \frac{v^{1/2}e^{-i\alpha v}}{\pi(1+v)^{1/2}}\times$$
$$\left[\int_{o}^{1}\frac{(1-w)^{1/2}e^{-i\alpha w}}{w^{1/2}(w+v)}\left\{2e^{i\alpha w} + 2i\alpha w\int_{-2w}^{\infty}\frac{e^{-i\alpha(t+w)}}{t+w}\,dt + F_1(w)\right\}dw\right] = 0$$

for $0 \leq v \leq 1$.

Now

$$\int_0^1 \frac{(1-w)^{1/2}}{w^{1/2}(w+v)}\,dw = \pi\left\{\frac{(1+v)^{1/2}}{v^{1/2}} - 1\right\} \tag{24}$$

for the range of v involved, as can be seen by the substitution $w = x^2/(1+x^2)$ for example (or from (E 10) as $t \to \infty$).

Also

$$\begin{aligned}&\int_0^1 \frac{(1-w)^{1/2}\,w^{1/2}\,e^{-i\alpha w}}{(w+v)} \int_{-2w}^{\infty} \frac{e^{-i\alpha(t+w)}}{t+w}\,dt\,dw \\ &= \int_0^1 \frac{(1-w)^{1/2}w^{1/2}}{w+v} \int_0^{\infty} \frac{e^{-i\alpha t}}{t-w}\,dt\,dw\end{aligned} \tag{25}$$

by replacing t by t-2w. Now the substitution $w = x^2/(1+x^2)$ shows that

$$\begin{aligned}\int_0^1 \frac{(1-w)^{1/2}\,w^{1/2}}{(w+v)(w-t)}\,dw = \frac{\pi}{t+v}\Big\{&v^{1/2}(1+v)^{1/2} \\ &+ t^{1/2}(t-1)^{1/2}\,H(t-1)\Big\} - \pi\end{aligned} \tag{26}$$

for $v \geq 0$, $t \geq 0$ (E11). Hence an inversion of the order of integration in (25) leads to

$$\begin{aligned}&\int_0^1 \frac{(1-w)^{1/2}w^{1/2}\,e^{-i\alpha w}}{w+v} \int_{-2w}^{\infty} \frac{e^{-i\alpha(t+w)}}{t+w}\,dt\,dw \\ &= \frac{i\pi}{\alpha}\left(e^{-i\alpha}-1\right) - \pi v^{1/2}(1+v)^{1/2}\int_0^{\infty} \frac{e^{-i\alpha t}}{t+v}\,dt \\ &+ \pi\int_1^{\infty}\left\{1 - \frac{t^{1/2}(t-1)^{1/2}}{t+v}\right\} e^{-i\alpha t}\,dt.\end{aligned}$$

Employing this result and (24) in (23) we obtain

$$\begin{aligned} F_1(v) = {} & \frac{2v^{1/2}e^{-i\alpha(v+1)}}{(1+v)^{1/2}} - \frac{2i\alpha v^{1/2}}{(1+v)^{1/2}}\, e^{-i\alpha v} \times \\ & \int_1^\infty \left\{ 1 - \frac{t^{1/2}(t-1)^{1/2}}{t+v} \right\} e^{-i\alpha t}\, dt - \frac{v^{1/2}e^{-i\alpha v}}{\pi(1+v)^{1/2}} \times \\ & \int_0^1 \frac{(1-w)^{1/2}F_1(w)\, e^{-i\alpha w}}{w^{1/2}(w+v)}\, dw . \end{aligned} \tag{27}$$

A rather simpler equation can be derived by deforming the contour of the first integral into the line joining 1 to $1-i\,\infty$. The branches of $t^{1/2}$ and $(t-1)^{1/2}$ are specified by taking $0 \geq \arg t \geq -\frac{1}{2}\pi$ and $0 \geq \arg\,(t-1) \geq -\frac{1}{2}\pi$. Then

$$\begin{aligned} F_1(v) = {} & \frac{2\alpha v^{1/2}}{(1+v)^{1/2}}\, e^{-\frac{1}{4}\pi i - i\alpha(1+v)} \times \\ & \int_0^\infty \frac{(1-it)^{1/2}t^{1/2}}{1+v-it}\, e^{-i\alpha t}\, dt - \frac{v^{1/2}e^{-i\alpha v}}{\pi(1+v)^{1/2}} \int_0^1 \frac{(1-w)^{1/2}F_1(w)e^{-i\alpha w}}{w^{1/2}(w+v)}\, dw. \end{aligned} \tag{28}$$

with the understanding that $0 \geq \arg(1-it) \geq -\frac{1}{2}\pi$. Putting

$$F_1(v) = \frac{v^{1/2}}{(1+v)^{1/2}}\, e^{-i\alpha v}\, I(v) \tag{29}$$

we have

$$\begin{aligned} I(v) = {} & 2\alpha e^{-i\alpha-\frac{1}{4}\pi i} \int_0^\infty \frac{(1-it)^{1/2}t^{1/2}}{1+v-it}\, e^{-\alpha t}\, dt \\ & - \frac{1}{\pi} \int_0^1 \frac{(1-w)^{1/2}\, I(w)e^{-2i\alpha w}}{(1+w)^{1/2}\,(w+v)}\, dw. \end{aligned} \tag{30}$$

The approximate solution of this when $\alpha \gg 1$ is considered in the next section

One might be tempted to solve (18) by iteration. However, if we write (18) as $F = F^{(o)} - KF$ and work in a Banach space with $||F|| = \max\limits_{0\leq x\leq 1} |F(x)|$ we have

$$||K|| = \max_{0\leq v\leq 1} \frac{v^{\frac{1}{2}}}{\pi(1+v)^{1/2}} \int_0^1 \left(\frac{1-t}{t}\right)^{1/2} \frac{dt}{t+v}$$

$$= \max_{0\leq v\leq 1} \left\{ 1 - \left(\frac{v}{1+v}\right)^{1/2} \right\} = 1$$

so that we cannot say that the iteration process will converge. Nevertheless the process may give an asymptotic development and this seems not unlikely in view of the equivalent form (30). This possibility will be examined in the next section.

4. Approximate Solution for Large α

Write (30) in the form

$$(31) \qquad I(v) = h_1(v) - \frac{1}{\pi} \int_0^1 \left(\frac{1-w}{1+w}\right)^{1/2} \frac{I(w)e^{-2i\alpha w}}{w+v} \, dw$$

where

$$(32) \qquad h_1(v) = 2\alpha e^{-i\alpha-\frac{1}{4}\pi i} \int_0^\infty \frac{(1-it)^{1/2} t^{1/2}}{1+v-it} e^{-\alpha t} dt.$$

A formal expansion by iteration immediately comes to mind and we examine the scheme

$$(33)\qquad I_1(v) = h_1(v),$$

$$(34)\qquad I_n(v) = -\frac{1}{\pi}\int_0^1 \left(\frac{1-w}{1+w}\right)^{1/2} \frac{I_{n-1}(w)e^{-2i\alpha w}}{w+v}\, dw.$$

The presence of the factor $e^{-2i\alpha w}$ suggests that the iterated terms will be of diminishing order of magnitude for large α. However, a check reveals that this is not so. To see this, note that as $\alpha \to \infty$

$$(35)\qquad h_1(v) \sim \frac{\pi^{1/2}\, e^{-i\alpha-\frac{1}{4}\pi i}}{\alpha^{1/2}(1+v)}$$

for all v in $[0,1]$. From this we observe that I_2, and I_3, ... are of increasing order of magnitude near $v = 0$. It is therefore necessary to adopt a less direct method of iteration, which eliminates the difficulty at the origin. Note that the difficulty cannot be evaded by subtracting a constant from I; this merely alters the order of magnitude for each iterate but they still increase. To avoid the trouble at the origin put

$$(36) \qquad I(v) = \left(\frac{1+v}{v}\right)^{1/2} \int_0^\infty \frac{H_1(t)}{t-v}\, e^{-2i\alpha(t-v)}\, dt + I_2(v)$$

where $H_1(t)$ will be specified shortly and I_2 is a new unknown (and not given by (33) and (34) which will be ignored from this point onwards). Since

$$(37) \qquad \int_0^1 \left(\frac{1-w}{w}\right)^{1/2} \frac{dw}{(w+v)(w-t)} = \frac{\pi}{t+v} \left\{ \left(\frac{t-1}{t}\right)^{1/2} H(t-1) - \left(\frac{1+v}{v}\right)^{1/2} \right\}$$

for $v > 0$ and $t > 0$ (the derivation is along the same lines as that for (26) or, alternatively, as in Appendix E (E10)) we have

$$I_2(v) = h_1(v) - \left(\frac{1+v}{v}\right)^{1/2} \int_0^\infty H_1(t) \left\{ \frac{e^{-2i\alpha(t-v)}}{t-v} - \right.$$

$$(38) \qquad \left. + \frac{e^{-2i\alpha t}}{t+v} \right\} dt + \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{H_1(t) e^{-2i\alpha t}}{t+v}\, dt$$

$$- \frac{1}{\pi} \int_0^1 \left(\frac{1-w}{1+w}\right)^{1/2} \frac{I_2(w) e^{-2i\alpha w}}{w+v}\, dw.$$

Now choose $H_1(t)$ so that

$$\left(\frac{1+v}{v}\right)^{1/2} \int_0^\infty H_1(t) \left\{ \frac{e^{-2i\alpha(t-v)}}{t-v} + \frac{e^{-2i\alpha t}}{t+v} \right\} dt = h_1(v)$$

or, equivalently, so that

$$\int_0^\infty H_1(t)\, e^{-i\alpha t} \left\{ \frac{e^{-i\alpha(t-v)}}{t-v} + \frac{e^{-i\alpha(t+v)}}{t+v} \right\} dt$$

$$= \left(\frac{v}{1+v}\right)^{1/2} h_1(v) e^{-i\alpha v}.$$

This equation is of the same form as (D1) and is satisfied, according to (D15), by taking

$$(39) \qquad H_1(y) = \frac{i\alpha}{4\pi} e^{i\alpha y} \int_0^\infty \left(\frac{x}{1+x}\right)^{1/2} h_1(x) e^{-i\alpha x} M(x,y)\, dx$$

where

$$(40) \quad M(x,y) = \frac{4xy\, H_1^{(2)}(\alpha x)\, J_0(\alpha y) - 4y^2 H_0^{(2)}(\alpha x) J_1(\alpha y)}{x^2 - y^2} .$$

With this choice of H_1 (38) reduces to

$$I_2(v) = h_2(v) - \frac{1}{\pi} \int_0^1 \left(\frac{1-w}{1+w}\right)^{1/2} \frac{I_2(w) e^{-2i\alpha w}}{w+v}\, dw$$

where

$$h_2(v) = \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{H_1(t) e^{-2i\alpha t}}{t+v}\, dt.$$

In view of the resemblance between (32) and the formula for $h_2(v)$ we now formulate the iteration scheme

$$(41) \quad I_n(v) = \left(\frac{1+v}{v}\right)^{1/2} \int_0^\infty \frac{H_n(t)}{t-v} e^{-2i\alpha(t-v)}\, dt, \qquad (n \geq 1)$$

$$(42) \quad H_n(y) = \frac{i\alpha}{4\pi} e^{i\alpha y} \int_0^\infty \left(\frac{x}{1+x}\right)^{1/2} h_n(x) e^{-i\alpha x} M(x,y)\, dx, \quad (n \geq 1)$$

$$(43)\quad h_n(v) = \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{H_{n-1}(t)e^{-2i\alpha t}}{t+v}\, dt, \quad (n \geq 2)$$

and h_1 is given by (32). We must now verify that the scheme has the desired properties.

Assume that $h_n(v)$ is regular in the portion $R(v) > -1$ of the complex v-plane and that $h_n(v)$ is bounded as $|v| \to \infty$ in that region; this is certainly true of $h_1(v)$. Now provide the integral in (42) with a complete contour from 0 to ∞ by adding an indentation below the pole $x = y$; to keep H_n unaltered the contribution of the indentation must be subtracted. The dominant behaviour of the Hankel functions at infinity in $0 \geq \arg x \geq -\frac{1}{2}\pi$ is given by a constant multiple of $e^{-i\alpha x}/x^{1/2}$. Therefore, when account is taken of the assumed properties of h_n, the complete contour just constructed can be deformed into the negative imaginary axis. Hence

$$H_n(y) = \frac{i}{\pi}\left(\frac{y}{1+y}\right)^{1/2} h_n(y) - \frac{2\alpha e^{\frac{1}{4}\pi i + i\alpha y}}{\pi^2} \int_0^\infty \left(\frac{x}{1-ix}\right)^{1/2} \frac{h_n(-ix)e^{-\alpha x}}{x^2+y^2} \times$$

$$(44)\qquad \left\{ xyK_1(\alpha x)J_0(\alpha y) - y^2K_0(\alpha x)J_1(\alpha y) \right\} dx$$

where $K_\nu(z)$ is the modified Bessel function related to the Hankel function by $\pi i H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i}) = -2e^{\frac{1}{2}\nu\pi i} K_\nu(z)$. The value of $(1-ix)^{1/2}$ is specified, as in the preceding section, by $0 \geq \arg(1-ix) \geq -\frac{1}{2}\pi$.

To evaluate (44) asymptotically use the integral representations

$$K_0(x) = \int_1^\infty \frac{e^{-xt}dt}{(t^2-1)^{1/2}}, \quad K_1(x) = x\int_1^\infty e^{-xt}(t^2-1)^{1/2}\, dt$$

Then the term involving K_1 in (44) can be written

$$-\frac{2\alpha^2}{\pi^2}\, y\, e^{\frac{1}{4}\pi i+i\alpha y}\, J_0(\alpha y)\int_1^\infty (t^2-1)^{1/2}\int_0^\infty \left(\frac{x^5}{1-ix}\right)^{1/2}\frac{h_n(-ix)}{x^2+y^2}\times$$

$$\cdot e^{-\alpha x(1+t)}\,dx\,dt.$$

The inner integral is a Laplace integral provided that h_n does not depend exponentially on α (which is certainly true of h_1). Thus, as $\alpha\to\infty$,

$$\int_0^\infty \left(\frac{x^5}{1-ix}\right)^{1/2}\frac{h_n(-ix)}{x^2+y^2}\, e^{-\alpha x(1+t)}\,dx \sim h_n(0)\times \tag{45}$$

$$\int_0^\infty \frac{x^{5/2}}{x^2+y^2}\, e^{-\alpha x(1+t)}\,dx$$

since $t\geq 1$. The integral on the right of (45) is a Gilbert integral and occurred in the theory of diffraction as early as 1863 [28]; it is expressible in terms of Lommel functions. However, it will mostly concern us only when $y\geq 1$. When $y\geq 1$ the asymptotic behaviour of (45) is easily found by replacing x^2+y^2 by y^2. Hence

$$\int_0^\infty \left(\frac{x^5}{1-ix}\right)^{1/2}\frac{h_n(-ix)}{x^2+y^2}\, e^{-\alpha x(1+t)}dx = \frac{15\pi^{1/2}\,h_n(0)}{8y^2\alpha^{7/2}(1+t)^{7/2}} + O\left(\frac{h_n}{\alpha^{9/2}}\right).$$

Since the substitution $t = 2\sec^2\theta - 1$ gives

$$\int_1^\infty \frac{(t^2-1)^{1/2}}{(t+1)^{7/2}}\, dt = 2^{-1/2}\int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta\, d\theta = \pi/2^{9/2}$$

the term involving K_1 in (44) supplies a contribution

$$-\frac{15\, h_n(0) e^{\frac{1}{4}\pi i + i\alpha y} J_0(\alpha y)}{\pi^{1/2}\alpha^{3/2} y\, 2^{13/2}} + O(\frac{h_n}{\alpha^{5/2}})\, J_0(\alpha y).$$

In fact this is valid not only when $y \geq 1$ but when $\alpha y >> 1$.

With regard to the term containing K_0, it can be written

$$\frac{2\alpha}{\pi^2}\, y^2 J_1(\alpha y)\, e^{\frac{1}{4}\pi i + i\alpha y} \int_1^\infty (t^2-1)^{-1/2} \int_0^\infty (\frac{x}{1-ix})^{1/2} \frac{h_n(-ix)}{x^2+y^2} \times$$

$$. e^{-\alpha x(1+t)}\, dx\, dt.$$

By a similar argument to that which has just been employed

$$\int_0^\infty (\frac{x}{1-ix})^{1/2} \frac{h_n(-ix)}{x^2+y^2}\, e^{-\alpha x(1+t)}\, dx$$

$$= \int_0^\infty \frac{x^{1/2}}{y^2}\left[h_n(0) + i\pi\left\{\frac{1}{2}\, h_n(0) - h_n'(0)\right\}\right] e^{-\alpha x(1+t)} dx + O(\frac{h_n}{\alpha^{7/2}})$$

$$= \frac{\pi^{\frac{1}{2}}}{2y^2\alpha^{3/2}(1+t)^{3/2}}\left[h_n(0) + \frac{3i}{2\alpha(1+t)}\left\{\frac{1}{2}\, h_n(0) - h_n'(0)\right\}\right] + O(\frac{h_n}{\alpha^{7/2}}).$$

The change of variable $t-1 = z(t+1)$ shows that

$$\int_1^\infty \frac{dt}{(t-1)^{1/2}(t+1)^n} = \frac{(n-\frac{3}{2})!\, \pi^{1/2}}{(n-1)!\, 2^{n-\frac{1}{2}}} .$$

Hence the contribution of the K_o term is

$$\frac{J_1(\alpha y)\quad e^{\frac{1}{4}\pi i+i\alpha y}}{\pi^{1/2}\alpha^{1/2}2^{5/2}}\left[h_n(0)+\frac{9i}{16\alpha}\left\{\frac{1}{2}h_n(0)-h_n'(0)\right\}\right]+O\left(\frac{h_n}{\alpha^{5/2}}\right)J_1(\alpha y).$$

Combining the two results we obtain

$$\begin{aligned} H_n(y) &= \frac{i}{\pi}\left(\frac{y}{1+y}\right)^{1/2} h_n(y)+\frac{J_1(\alpha y)\quad e^{\frac{1}{4}\pi i+i\alpha y}}{\pi^{1/2}\alpha^{1/2}2^{5/2}} \\ &\cdot\left[h_n(0)+\frac{9i}{16\alpha}\left\{\frac{1}{2}h_n(0)-h_n'(0)\right\}\right] \\ &-\frac{15h_n(0)e^{\frac{1}{4}\pi i+i\alpha y}\quad J_o(\alpha y)}{\pi^{1/2}\alpha^{3/2}y\;2^{13/2}} \\ &+O\left(\frac{h_n}{\alpha^{5/2}}\right)J_o(\alpha y) \end{aligned} \tag{46}$$

when $\alpha y \gg 1$. For $\alpha y \gg 1$ the Bessel functions can be replaced by their asymptotic expansions, namely

$$J_o(\alpha y) = \left(\frac{2}{\pi\alpha y}\right)^{1/2}\cos(\alpha y-\tfrac{1}{4}\pi)+O(\alpha^{-\frac{3}{2}}),$$

$$J_1(\alpha y) = \left(\frac{2}{\pi\alpha y}\right)^{1/2}\left\{\cos(\alpha y-\tfrac{3}{4}\pi)-\frac{3}{8\alpha y}\sin(\alpha y-\tfrac{3}{4}\pi)\right\}+O(\alpha^{-\frac{5}{2}}).$$

Then, for $\alpha y >> 1$,

$$H_n(y) = \frac{i}{\pi}\left(\frac{y}{1+y}\right)^{1/2} h_n(y) - \frac{1}{\pi\alpha y^{1/2}}\left\{(e^{2i\alpha y}-i)h_{n,o} + \frac{9h_n(0)}{128\alpha y}(e^{2i\alpha y}+i)\right\} + O\left(\frac{h_n}{\alpha^3}\right) \tag{47}$$

where

$$h_{n,o} = \frac{i}{8}\left[h_n(0) + \frac{9i}{16\alpha}\left\{\frac{1}{2}h_n(0) - h_n'(0)\right\}\right]$$

We can now see from (43) (with $n + 1$ for n) that h_{n+1} has those properties of h_n that were used in deducing (47). Hence there is a corresponding formula for H_{n+1}. Since h_1 has these properties it follows, by induction, that (47) is valid for all n. Furthermore, the dominant contribution of (47) to $h_{n+1}(v)$ for large α is $O(h_{n,o}\alpha^{-1})$. Thus, in the relevant range, $h_{n+1}(v)$ is smaller than $h_n(v)$ by a factor of $O(\alpha^{-1})$ and hence, by (47), H_{n+1} is smaller than H_n by a similar factor. Consequently, the iteration scheme is appropriate for large α.

5. Determination of the Distribution on the Disc

Denote the contribution of $F(v)-F_1(v)$ to f by f_o and the contribution of I_n to f by f_n. Then, from (14), (22) and (29),

$$f_o(w) = - \frac{2e^{i\alpha w}}{\pi^2 w^{1/2} \, (1-w)^{1/2}} \int_0^1 \frac{(1-v)^{1/2} e^{-i\alpha v}}{v^{1/2}(v-w)} \tag{48}$$

$$\cdot \left\{ e^{i\alpha v} + i\alpha v \int_{-2v}^{\infty} \frac{e^{-i\alpha(t+v)}}{t+v} \, dt \right\} dv ,$$

$$f_n(w) = \frac{-e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \left(\frac{1-v}{1+v}\right)^{1/2} \times \tag{49}$$

$$\frac{I_n(v) \, e^{-2i\alpha v}}{v-w} \, dv \quad (n \geq 1).$$

Since (E5)

$$\int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{v-w} = -\pi \quad (0 < w < 1),$$

$$f_o(w) = \frac{2e^{i\alpha w}}{\pi w^{1/2}(1-w)^{1/2}} \left\{ 1 - \frac{i\alpha}{\pi} \int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{v-w} \times \right.$$

$$\left. \cdot \int_0^{\infty} \frac{e^{-i\alpha t}}{t-v} \, dt \, dv \right\} .$$

Employing (E7)

$$\int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{(v-w)(v-t)} \, dv = -\pi + \pi^2 w^{1/2} (1-w)^{1/2} \delta(t-w)$$

$$+ \frac{\pi t^{1/2}(t-1)^{1/2}}{t-w} \, H(t-1) \quad (0 < w < 1, \; t > 0)$$

we have

$$f_o(w) = \frac{2e^{i\alpha w}}{\pi w^{1/2}(1-w)^{1/2}}\left[e^{-i\alpha} + i\alpha\pi w^{1/2}(1-w)^{1/2}e^{-i\alpha w}\right.$$

$$\left. + i\alpha\int_1^\infty\left\{\frac{t^{1/2}(t-1)^{1/2}}{t - w} - 1\right\}e^{-i\alpha t}\,dt\right].$$

A slightly simpler form is obtained by deforming the path of integration into the line joining 1 to 1-i ∞. Then

$$(50)\quad f_o(w) = 2i\alpha + \frac{2\alpha e^{i\alpha(w-1)-\frac{1}{4}\pi i}}{\pi w^{1/2}(1-w)^{1/2}}\int_0^\infty \frac{(1-it)^{1/2}t^{1/2}}{1 - w - it}e^{-\alpha t}\,dt$$

where, as before, $0 \geq \arg(1-it) \geq -\frac{1}{2}\pi$.

With regard to f_n substitute in (49) for I_n from (41). Then

$$f_n(w) = \frac{-e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}}\int_0^1(\frac{1-v}{v})^{1/2}\frac{1}{v-w}\int_0^\infty\frac{H_n(t)e^{-2i\alpha t}}{t-v}\,dt\,dv.$$

Hence, using (E3)

$$\int_0^1(\frac{1-v}{v})^{1/2}\frac{dv}{(v-w)(v-t)} = \pi^2(\frac{1-w}{w})^{1/2}\delta(t-w)$$

$$+\pi(\frac{t-1}{t})^{1/2}\frac{H(t-1)}{t - w}\quad (0 < w < 1,\ t > 0),$$

we find

$$f_n(w) = \frac{e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}}\left[\pi^2(\frac{1-w}{w})^{1/2}H_n(w)\,e^{-2i\alpha w} + \right.$$

$$+ \pi \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{H_n(t)e^{-2i\alpha t}}{t - w} \, dt \Big]$$

(51)
$$= \frac{H_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{1/2}(1-w)^{1/2}} \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \times$$

$$\cdot \frac{H_n(t)e^{-2i\alpha t}}{t - w} \, dt.$$

So far the analysis is exact. Now consider what happens when α is large. The first two terms of the asymptotic development of the field f_o are given by

$$f_o(w) \sim 2i\alpha + \frac{2\alpha e^{i\alpha(w-1)-\frac{1}{4}\pi i}}{\pi w^{1/2}(1-w)^{1/2}} \int_0^\infty \frac{t^{1/2}e^{-\alpha t}}{1-w-it} \, dt,$$

t not being replaced by zero in the denominator since 1-w vanishes at the edge of the disc $w = 1$. However, if $\beta > -1$ and $v > 0$,

(52)
$$\int_0^\infty \frac{t^\beta e^{-\alpha t}}{v-it} \, dt = i \int_0^\infty t^\beta e^{-\alpha t} \int_0^\infty e^{-x(t+iv)} dx \, dt$$

$$= \beta! \, i \int_0^\infty \frac{e^{-ivx}}{(x+\alpha)^{\beta+1}} \, dx$$

$$= \beta! \, i \, e^{i\alpha v} \int_\alpha^\infty \frac{e^{-ivx}}{x^{\beta+1}} \, dx .$$

Hence, taking $\beta = \frac{1}{2}$, we have

$$f_o(w) \sim 2i\alpha + \frac{\alpha \, e^{\frac{1}{4}\pi i}}{\pi^{1/2} w^{1/2}(1-w)^{1/2}} \int_\alpha^\infty \frac{e^{-i(1-w)x}}{x^{3/2}} \, dx. \quad (53)$$

It is easily confirmed that this gives the same distribution near the edge as if it were assumed that such a region behaved as a semi-infinite plane, with edge along the tangent, placed in the incident field. The f_n and terms omitted in the asymptotic expression (53) represent corrections to this point of view.

For f_n the formula (47) may be employed in the integral of (51). The result is

$$f_n(w) \sim \frac{H_n(w)e^{-i\alpha w}}{w} + \frac{h_n(1)e^{-i\alpha w+\frac{1}{4}\pi i}}{2^{3/2}\pi^{3/2}w^{1/2}(1-w)^{1/2}} \times \tag{54}$$

$$\int_{2\alpha}^{\infty} \frac{e^{-i(1-w)x}}{x^{3/2}}\,dx - \frac{h_{n,o}e^{i\alpha w}}{\pi\alpha w^{3/2}(1-w)^{1/2}}\left\{1-(1-w)^{1/2}\right\}$$

the error being of order $h_n\,\alpha^{-3/2}$. Unless w is near the origin (47) can be employed in the first term of (54) with the consequence

$$f_n(w) \sim \frac{ih_n(w)e^{-i\alpha w}}{\pi w^{1/2}(1+w)^{1/2}} + \frac{h_n(1)e^{-i\alpha w+\frac{1}{4}\pi i}}{2^{3/2}\,\pi^{3/2}\,w^{1/2}(1-w)^{1/2}} \times \tag{55}$$

$$\cdot \int_{2\alpha}^{\infty} \frac{e^{-i(1-w)x}}{x^{3/2}}\,dx + \frac{h_{n,o}}{\pi\alpha w^{3/2}}\left\{ie^{-i\alpha w} - \frac{e^{i\alpha w}}{(1-w)^{1/2}}\right\} .$$

The resemblance of the second term of (55) to the second term of (53) will be noted. In essence, it represents the effect of waves which have been multiply diffracted (n times) at diametrically opposite points of the edge.

When w is near the origin it is no longer to replace $H_n(w)$ in (54) by its expansion (47) and so (55) does not hold.

Suppose that $\alpha w < < 1$; then (44) shows that

$$H_n(w) = \frac{i}{\pi} w^{1/2} h_n(0) - \frac{2\alpha w e^{\frac{1}{4}\pi i}}{\pi^2} \int_0^\infty \left(\frac{x^3}{1-ix}\right)^{1/2} \times$$

$$\cdot \frac{h_n(-ix)e^{-\alpha x}}{x^2 + w^2} K_1(\alpha x)\, dx + O(w)$$

$$= \frac{i}{\pi} w^{1/2} h_n(0) - \frac{2\alpha w e^{\frac{1}{4}\pi i}}{\pi^2} \int_0^\epsilon \left(\frac{x^3}{1-ix}\right)^{1/2} \frac{h_n(-ix)}{x^2 + w^2} \times$$

$$\cdot K_1(\alpha x) dx + O(w)$$

where $1 > > \alpha\, \epsilon > > \alpha\, w$. Because of the small range of integration we can approximate $K_1(\alpha\, x)$ by $1/\alpha x$, $h_n(-ix)$ by $h_n(0)$, $e^{-\alpha x}$ by 1 and $1-ix$ by 1. Hence

$$H_n(w) \approx \frac{i}{\pi} w^{1/2} h_n(0) - \frac{2we^{\frac{1}{4}\pi i}}{\pi^2} h_n(0) \int_0^\epsilon \frac{x^{1/2}}{x^2 + w^2}\, dx$$

$$\approx \frac{i}{\pi} w^{1/2} h_n(0) - \frac{2w^{1/2} e^{\frac{1}{4}\pi i}}{\pi^2} h_n(0) \int_0^{\epsilon/w} \frac{x^{1/2}}{x^2 + 1}\, dx$$

so that $H_n(w) \approx \frac{i}{\pi} w^{1/2} h_n(0) \left\{1 - \frac{2e^{-\frac{1}{4}\pi i}}{\pi} \int_0^\infty \frac{x^{1/2}}{x^2+1}\, dx\right\}$

(56)

$$\approx -\frac{1}{\pi} w^{1/2} h_n(0)$$

since $\int_0^\infty \frac{x^{1/2} dx}{x^2 + 1} = 2^{-\frac{1}{2}}\pi$. Consequently, for $\alpha\, w < < 1$,

$$(57)\qquad f_n(w) \approx -\frac{h_n(0)}{\pi w^{1/2}} + \frac{h_n(1)e^{-\frac{1}{4}\pi i}e^{-2i\alpha}}{2^3\pi^{3/2}w^{1/2}\alpha^{3/2}} - \frac{h_{n,o}}{2\pi\alpha w^{1/2}} .$$

There is no difficulty in confirming that the first term of f_1 when $\alpha w << 1$ cancels the last term of f_o in (53) leaving a singularity at $w = 0$ which is smaller as $\alpha \to \infty$ than that in f_o. Similarly, the addition of f_2 produces a singularity of smaller magnitude than that in $f_o + f_1$.

It is now possible to obtain an explicit expression for the distribution on the disc correct to any order desired. All that is required is to calculate integrals asymptotically to as many terms as are necessary. This calculation will be omitted here since generally most interest centers on various integrals of the distribution. Some of these integrals will be examined in the following sections.

6. The High Frequency Scattering Coefficient

It is known (see, for example, Jones [29]) that the scattering coefficient of the circular disc in an incident plane wave is $-(4/ka^2)\mathcal{I}A$ when unnormalised coordinates are used and A is the amplitude of the scattered wave in the direction of propagation of the incident wave. Hence, in terms of our variables, the scattering coefficient σ is given by

$$(58)\qquad \sigma = \frac{2}{\alpha}\mathcal{I}\int_o^1 wf(w)\,dw.$$

Devote by σ_n the contribution to σ by f_n. Then, from (53),

$$\sigma_o \sim \frac{2}{\alpha} \mathcal{J} \left[i\alpha + \frac{\alpha e^{\frac{1}{4}\pi i}}{\pi^{1/2}} \int_o^1 \frac{w^{1/2}}{(1-w)^{1/2}} \int_\alpha^\infty \frac{e^{-i(1-w)x}}{x^{3/2}} dx \, dw \right].$$

However, more terms in the asymptotic expansion will be required and to derive this we use (50) to give further terms in the expansion of f_o. It is a straight forward matter to show that

$$f_o(w) \sim 2i\alpha + \frac{\alpha e^{\frac{1}{4}\pi i}}{\pi^{1/2} w^{1/2} (1-w)^{1/2}} . \int_\alpha^\infty \frac{e^{-i(1-w)x}}{x^{3/2}} \times$$

$$\left\{ 1 - \frac{3i}{4x} + \frac{15}{32x^2} + \frac{105i}{128x^3} \right\} dx.$$

Now, for $x \geq \alpha$,

(59)

$$\int_o^1 \frac{w^{1/2} e^{iwx}}{(1-w)^{1/2}} dw \sim \frac{\pi^{1/2} e^{ix - \frac{1}{4}\pi i}}{x^{1/2}} \left(1 + \frac{i}{4x} + \frac{3}{32x^2} - \frac{15i}{128x^3} \right)$$

$$+ \frac{\pi^{1/2} e^{\frac{3}{4}\pi i}}{2x^{3/2}} \left(1 + \frac{3i}{4x}\right).$$

Hence

$$\sigma_o \sim \frac{2}{\alpha} \mathcal{J} \left[i\alpha + \alpha \int_\alpha^\infty \left\{ \frac{1}{x^2} \left(1 - \frac{3i}{4x} + \frac{15}{32x^2} + \frac{105i}{128x^3}\right) \times \right.\right.$$

$$\left.\left. \cdot \left(1 + \frac{i}{4x} + \frac{3}{32x^2} - \frac{15i}{128x^3}\right) - \frac{e^{-ix}}{2x^3} \right\} dx \right]$$

$$\sim \frac{2}{\alpha}\mathcal{I}\left[i\alpha + \alpha\left(\frac{1}{\alpha} - \frac{i}{4\alpha^2} + \frac{1}{4\alpha^3} + \frac{3i}{16\alpha^4}\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{ie^{-i\alpha}}{\alpha^2} - \frac{3e^{-i\alpha}}{2\alpha^3}\right)\right]$$

$$(60) \qquad \sim 2\left\{1 - \frac{1}{4\alpha^2} + \frac{1}{2\alpha^3}\cos\alpha + \frac{3}{16\alpha^4} + \frac{3}{2\alpha^4}\sin\alpha\right\}$$

the error being $O(\alpha^{-5})$.

The expression for σ_n $(n \geq 1)$ is, from (51),

(61)

$$\sigma_n = \frac{2}{\alpha}\mathcal{I}\left[\int_0^1 H_n(w)e^{-i\alpha w}\,dw + \frac{1}{\pi}\int_0^1\left(\frac{w}{1-w}\right)^{1/2} e^{i\alpha w} \times\right.$$

$$\left.\int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{H_n(t)e^{-2i\alpha t}}{t-w}\,dt\,dw\right].$$

Now

$$\int_0^1 H_n(w)e^{-i\alpha w}\,dw = \int_0^\infty H_n(w)e^{-i\alpha w}\,dw - \int_1^\infty H_n(w)e^{-i\alpha w}\,dw.$$

Also

$$(62) \qquad \int_0^\infty H_n(w)e^{-i\alpha w}\,dw = \frac{i\alpha}{4\pi}.\int_0^\infty\int_0^\infty \left(\frac{x}{1+x}\right)^{1/2} h_n(x)e^{-i\alpha x} M(x,w)\,dx\,dw.$$

To evaluate the integrals with respect to w consider

$$\int_D \frac{z^{\nu+1}H_\nu^{(1)}(\alpha z)}{z^2 - y^2}\,dz$$

where $-1 < \mathcal{R}(\nu) < \frac{3}{2}$, $\alpha > 0$, $y > 0$ and D is a contour along the entire real axis apart from indentations above $z = 0, \pm y$. On account of the asymptotic behaviour of the Hankel function and the range of ν the contour can be deformed to infinity in the positive imaginary part of the complex plane to give

$$\int_D \frac{z^{\nu+1} H_\nu^{(1)}(\alpha z)}{z^2 - y^2} dz = 0.$$

In view of the range of ν the contribution of the indentation about the origin vanishes as its radius shrinks to zero. The contributions from the simple poles $z = \pm y$ one calculated in the usual way and

$$\int_{-\infty}^{\infty} \frac{x^{\nu+1} H_\nu^{(1)}(\alpha x)}{x^2 - y^2} dx$$

$$= \frac{1}{2}\pi i y^\nu \{H_\nu^{(1)}(\alpha y) - e^{(\nu+1)\pi i} H_\nu^{(1)}(\alpha y e^{\pi i})\} .$$

Since $H_\nu^{(1)}(xe^{\pi i}) = e^{-\nu\pi i} H_\nu^{(1)}(x) - 2e^{-\nu\pi i} J_\nu(x)$ we obtain

$$(63) \qquad \int_0^{\infty} \frac{x^{\nu+1} J_\nu(\alpha x)}{x^2 - y^2} dx = - \frac{1}{2}\pi y^\nu Y_\nu(\alpha y)$$

for $-1 < \Re(\nu) < \frac{3}{2}$, $y > 0$, $\alpha > 0$. In particular,

$$(64) \qquad \int_0^{\infty} \frac{w J_0(\alpha w)}{x^2 - w^2} dw = \frac{1}{2} \pi \; Y_0(\alpha x)$$

and

$$\int_0^\infty \frac{w^2 J_1(\alpha w)}{x^2 - w^2}\, dw = \frac{1}{2}\pi x Y_1(\alpha x). \tag{65}$$

The substitution of (64) and (65) in (62) gives, after a use of the Wronskian relation,

$$\int_0^\infty H_n(w)e^{-i\alpha w}dw = \frac{i}{\pi}\int_0^\infty \left(\frac{x}{1+x}\right)^{1/2} h_n(x)e^{-i\alpha x}\, dx\,. \tag{66}$$

In $\int_1^\infty H_n(w)e^{-i\alpha w}\, dw$ the formula (47) can be employed so that

$$\int_1^\infty H_n(w)e^{-i\alpha w}\, dw = \int_1^\infty \Big\{\frac{i}{\pi}\left(\frac{x}{1+x}\right)^{1/2} h_n(x)e^{-i\alpha x} - \frac{h_{n,o}}{\pi\alpha x^{1/2}} \times$$

$$\cdot\left(e^{i\alpha x} - ie^{-i\alpha x}\right)\Big\}\, dx + O\left(\frac{h_n}{\alpha^3}\right) \cdot$$

Combining this with (66) we have

$$\begin{aligned}\int_0^1 H_n(w)e^{-i\alpha w}\, dw &= \frac{i}{\pi}\int_0^1 \left(\frac{x}{1+x}\right)^{1/2} h_n(x)e^{-i\alpha x}\, dx \\ &\quad + \frac{h_{n,o}}{\pi\alpha}\int_1^\infty \frac{1}{x^{1/2}}\left(e^{i\alpha x} - ie^{-i\alpha x}\right)dx + O\left(\frac{h_n}{\alpha^3}\right) \cdot\end{aligned} \tag{67}$$

The asymptotic expansion of the first term of (67) can be deduced from theorems about Fourier transforms or, alternatively, the contour of integration may be deformed into the negative imaginary axis and the straight line joining 1 to 1-i∞. In the latter case

$$\frac{i}{\pi}\int_0^1\left(\frac{x}{1+x}\right)^{\frac{1}{2}} h_n(x)e^{-i\alpha x}dx = \frac{e^{-\frac{1}{4}\pi i}}{\pi}\int_0^\infty\left(\frac{x}{1-ix}\right)^{\frac{1}{2}} h_n(-ix)e^{-\alpha x}\,dx$$

$$-\frac{e^{-i\alpha}}{\pi}\int_0^\infty\left(\frac{1-ix}{2-ix}\right)^{\frac{1}{2}} h_n(1-ix)e^{-\alpha x}\,dx$$

$$(68)\qquad = \frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left[h_n(0)+\frac{3i}{2\alpha}\left\{\frac{1}{2}h_n(0)-h_n'(0)\right\}\right]$$

$$-\frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\left[h_n(1)-\frac{i}{\alpha}\left\{\frac{1}{4}h_n(1)+h_n'(1)\right\}\right]+O\left(\frac{h_n}{\alpha^3}\right).$$

Furthermore

$$\int_1^\infty\frac{e^{i\alpha x}}{x^{\frac{1}{2}}}dx\sim\frac{ie^{i\alpha}}{\alpha},\quad\int_1^\infty\frac{e^{-i\alpha x}}{x^{\frac{1}{2}}}dx\sim-\frac{ie^{-i\alpha}}{\alpha}.$$

Using these results, together with (68), in (67) we obtain

$$\int_0^1 H_n(w)e^{-i\alpha w}\,dw = \frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left[h_n(0)+\frac{3i}{2\alpha}\left\{\frac{1}{2}h_n(0)-h_n'(0)\right\}\right]$$

$$(69)\qquad -\frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\left[h_n(1)-\frac{i}{\alpha}\left\{\frac{1}{4}h_n(1)+h_n'(1)\right\}\right]$$

$$-\frac{h_n(0)}{8\pi\alpha^2}\left(e^{i\alpha}+ie^{-i\alpha}\right)+O\left(\frac{h_n}{\alpha^3}\right)$$

since $h_{n,o}$ can be replaced by $ih_n(0)/8$ to the order to which (69) is accurate. This completes the treatment of the first integral in (61).

With regard to the second integral of (61) we can use (47) for $H_n(t)$ since only the range $t \geq 1$ is involved. Hence the asymptotic evaluation of

$$\frac{1}{\pi}\int_0^1 \left(\frac{w}{1-w}\right)^{1/2} e^{i\alpha w} \int_1^\infty \left(\frac{t-1}{t}\right)^{1/2} \frac{e^{-2i\alpha t}}{t-w} \left[\frac{i}{\pi}\left(\frac{t}{1+t}\right)^{1/2} h_n(t) \right.$$

(70)

$$\left. - \frac{1}{\pi\ \alpha t^{\frac{1}{2}}} \left\{ (e^{2i\alpha t} - i) h_{n,o} + \frac{9h_n(o)}{128\alpha t}(e^{2i\alpha t} + i) \right\} \right] dt\, dw,$$

is required. Now

$$\int_0^1 \left(\frac{w}{1-w}\right)^{1/2} \frac{e^{i\alpha w}}{t - w}\, dw = e^{\frac{3}{4}\pi i} \int_0^\infty \left(\frac{x}{1-ix}\right)^{1/2} \frac{e^{-\alpha x}}{t-ix}\, dx$$

$$-e^{\frac{3}{4}\pi i} \int_0^\infty \left(\frac{1+ix}{x}\right)^{1/2} \frac{e^{i\alpha - \alpha x}}{t-1-ix}\, dx$$

$$\sim \pi^{1/2} e^{\frac{1}{4}\pi i + i\alpha t} \int_\alpha^\infty \frac{e^{-i(t-1)x}}{x^{\frac{1}{2}}} \times$$

$$\cdot \left(1 + \frac{i}{4x} - \frac{e^{-ix}}{2x}\right) dx$$

from (52). Consequently the terms in the bracket [ ] of (70) which do not involve $e^{2i\alpha t}$ have asymptotic behaviour

$$\frac{e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\int_{\alpha}^{\infty}\frac{e^{ix}}{x^{\frac{1}{2}}}\left(1+\frac{i}{4x}-\frac{e^{-ix}}{2x}\right)\int_{1}^{\infty}(t-1)^{1/2}\,e^{-it(x+\alpha)}\times$$

$$\left[\frac{ih_n(t)}{\pi(1+t)^{1/2}}+\frac{i}{\pi\alpha t}\left\{h_{n,o}-\frac{9h_n(o)}{128\alpha t}\right\}\right]dt\,dx$$

$$\sim\frac{e^{-i\alpha}}{2^{\frac{3}{2}}\pi}\int_{\alpha}^{\infty}\frac{1}{x^{\frac{1}{2}}(x+\alpha)^{\frac{3}{2}}}\left(1+\frac{i}{4x}-\frac{e^{-ix}}{2x}\right)\left[h_n(1)+\frac{3i}{2(\alpha+x)}\times\right.$$

$$\left.\left\{\frac{1}{4}h_n(1)-h_n'(1)\right\}+2^{\frac{1}{2}}\frac{h_{n,o}}{\alpha}\right]dx.$$

The change of variable $x = \alpha\tan^2\theta$ shows that

$$\int_{\alpha}^{\infty}\frac{dx}{x^{\frac{1}{2}}(x+\alpha)^{\frac{3}{2}}}=\frac{2}{\alpha}\left(1-\frac{1}{2^{\frac{1}{2}}}\right),$$

$$\int_{\alpha}^{\infty}\frac{dx}{x^{\frac{1}{2}}(x+\alpha)^{5/2}}=\frac{2}{3\alpha^{2}}\left(2-\frac{5}{2^{\frac{3}{2}}}\right),$$

$$\int_{\alpha}^{\infty}\frac{dx}{x^{\frac{3}{2}}(x+\alpha)^{\frac{3}{2}}}=\frac{2}{\alpha^{2}}\left(\frac{3}{2^{\frac{1}{2}}}-2\right).$$

Hence our asymptotic formula becomes

(71)

$$\frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\Big[(1-\frac{1}{2^{\frac{1}{2}}})h_n(1)+\frac{i}{4\alpha}(\frac{3}{2^{\frac{1}{2}}}-2)h_n(1)$$

$$+\frac{i}{2\alpha}\{\frac{1}{4}h_n(1)-h_n'(1)\}(2-\frac{5}{2^{\frac{3}{2}}})+\frac{2^{\frac{1}{2}}i}{8\alpha}h_n(o)(1-\frac{1}{2^{\frac{1}{2}}})\Big]$$

$$=\frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\Big[(1-\frac{1}{2^{\frac{1}{2}}})h_n(1)+\frac{i}{4\alpha}(\frac{7}{2^{\frac{5}{2}}}-1)h_n(1)$$

$$-\frac{i}{\alpha}(1-\frac{5}{2^{\frac{5}{2}}})h_n'(1)+\frac{ih_n(o)}{8\alpha}(2^{\frac{1}{2}}-1)\Big]$$

terms of $O(h_n/\alpha^3)$ being neglected.

The remaining term of (70) is

$$-\frac{1}{\pi^2\alpha}\int_o^1(\frac{w}{1-w})^{1/2}e^{i\alpha w}\int_1^\infty\frac{(t-1)^{1/2}}{t(t-w)}\{h_{n,o}+\frac{9h_n(o)}{128\alpha t}\}\,dt\,dw.$$

For $w \leq 1$,

$$\int_1^\infty\frac{(t-1)^{1/2}}{t(t-w)}dt=\frac{\pi}{w}\{1-(1-w)^{1/2}\}$$

and

$$\int_1^\infty \frac{(t-1)^{1/2}}{t^2(t-w)}\,dt = \frac{\pi}{w^2}\left\{1 - \frac{1}{2}w - (1-w)^{1/2}\right\}$$

as may be confirmed by the change of variable $z = (t-1)^{1/2}/(1+t^{1/2})$. Hence our integral is

$$(72) \qquad -\frac{h_{n,o}}{\pi\alpha}\int_0^1 \frac{e^{i\alpha w}}{w^{\frac{1}{2}}}\left\{\frac{1}{(1-w)^{1/2}} - 1\right\}dw - \frac{9h_n(o)}{128\pi\alpha^2}\times$$

$$\int_0^1 \frac{e^{i\alpha w}}{w^{\frac{3}{2}}(1-w)^{1/2}}\left\{1 - \frac{1}{2}w - (1-w)^{1/2}\right\}dw$$

$$\sim \frac{e^{\frac{3}{4}\pi i}\,h_{n,o}}{\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left\{e^{i\alpha}\left(1 - \frac{i}{4\alpha}\right) - \frac{1}{4\alpha} - \frac{e^{i\alpha-\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\right\} + \frac{9h_n(o)\,e^{\frac{3}{4}\pi i+i\alpha}}{256\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}}$$

terms of $O(h_n/\alpha^3)$ being ignored.

Collecting together (69), (71) and (72) we obtain, from (61),

$$\sigma_n = \frac{2}{\alpha}\mathscr{I}\Bigg[\frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left\{h_n(0) + \frac{3i}{2\alpha}\left(\frac{1}{2}h_n(0) - h_n'(0)\right)\right\} - \frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\Big\{h_n(1)$$

$$- \frac{i}{\alpha}\left(\frac{1}{4}h_n(1) + h_n'(1)\right)\Big\} - \frac{h_n(0)}{8\pi\alpha^2}\left(e^{i\alpha} + ie^{-i\alpha}\right)$$

$$+ \frac{e^{-i\alpha}}{2^{\frac{1}{2}}\pi\alpha}\Big\{(1- \frac{1}{2^{\frac{1}{2}}})h_n(1) + \frac{i}{4\alpha}(\frac{7}{2^{\frac{5}{2}}} - 1)h_n(1) - \frac{i}{\alpha}(1 - \frac{5}{2^{\frac{5}{2}}})h_n'(1)$$

$$+ \frac{ih_n(0)}{8\alpha}\,(2^{\frac{1}{2}} - 1)\Big\} + \frac{e^{\frac{3}{4}\pi i}h_{n,o}}{\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\Big\{e^{i\alpha}(1- \frac{i}{4\alpha})$$

$$- \frac{1}{4}\,\alpha - \frac{e^{i\alpha - \frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\Big\} + \frac{9h_n(0)\,e^{\frac{3}{4}\pi i + i\alpha}}{256\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}}\Bigg] + O(\frac{h_n}{\alpha^4})$$

$$= \frac{2}{\alpha}\Im\Bigg[- \frac{e^{-i\alpha}}{2\pi\alpha}\Big\{h_n(1) - \frac{7i}{16\alpha}\,h_n(1) - \frac{5i}{4\alpha}\,h_n'(1) + \frac{2^{\frac{1}{2}}ih_n(0)}{8\alpha}\Big\} + \frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\Big\{h_n(o)$$

$$+ \frac{i}{\alpha}\Big(\frac{13}{16}\,h_n(o) - \frac{3}{2}h_n'(o)\Big)\Big\} - \frac{e^{\frac{1}{4}\pi i + i\alpha}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\Big\{h_n(0)$$

$$- \frac{i}{4\alpha}(h_n(0) + \frac{9}{4}h_n'(0))\Big\}\Bigg] + O(\frac{h_n}{\alpha^4}). \tag{73}$$

All that remains to be done is to find asymptotic formulae for h_n. This can be achieved in a comparatively straight forward manner. For, when $n \geq 2$, the combination of (43) and (47) gives

$$h_n(v) = \int_1^\infty (\frac{t-1}{t})^{1/2}\,\frac{e^{-2i\alpha t}}{t+v}\Big[\frac{i}{\pi}(\frac{t}{1+t})^{1/2}\,h_{n-1}(t) - \frac{1}{\pi\alpha t^{\frac{1}{2}}}\{(e^{2i\alpha t} - i)h_{n-1,o} +$$

$$+ \frac{9h_{n-1}(o)}{128\alpha t}(e^{2i\alpha t} + i)\}\Big]\, dt + O\Big(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\Big) = \frac{e^{-2i\alpha-\frac{1}{4}\pi i}h_{n-1}(1)}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}(1+v)}$$

$$+ \frac{h_{n-1,o}}{\alpha v}\{1-(1+v)^{1/2}\} - \frac{9h_{n-1}(0)}{128\alpha^{2}v^{2}}\{1+\frac{1}{2}v$$

$$- (1+v)^{1/2}\} + O\Big(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\Big).$$

Hence

(74)
$$h_n(1) = \frac{h_{n-1,o}}{\alpha}(1-2^{\frac{1}{2}}) + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}h_{n-1}(1) - \frac{9h_{n-1}(o)}{128\alpha^{2}}\times$$

$$(\frac{3}{2} - 2^{\frac{1}{2}}) + O\Big(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\Big),$$

(75)
$$h_n'(1) = \frac{h_{n-1,o}}{\alpha}\Big(\frac{3}{2^{\frac{3}{2}}} - 1\Big) - \frac{e^{-2i\alpha-\frac{1}{4}\pi i}h_{n-1}(1)}{32\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} - \frac{9h_{n-1}(o)}{256\alpha^{2}}\Big(\frac{7}{2^{\frac{1}{2}}} - 5\Big)$$

$$+ O\Big(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\Big),$$

$$(76)\qquad h_n(0) = -\frac{h_{n-1,o}}{2\alpha} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) + \frac{9h_{n-1}(0)}{1024\alpha^2} + O\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right),$$

$$(77)\qquad h_n'(0) = \frac{h_{n-1,o}}{8\alpha} - \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) + \frac{9h_{n-1}(0)}{2048\alpha^2} + O\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right).$$

The corresponding formulae for h_1 are obtained from (32). Thus

$$h_1(v) = \frac{\pi^{\frac{1}{2}} e^{-i\alpha-\frac{1}{4}\pi i}}{\alpha^{1/2}(1+v)} \left\{1 + \frac{3i(1-v)}{4\alpha(1+v)}\right\} + O(\alpha^{-\frac{5}{2}})$$

so that

$$h_1(1) = \frac{\pi^{\frac{1}{2}}}{2\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{5}{2}}),$$

$$h_1'(1) = -\frac{\pi^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}),$$

$$(78)\qquad h_1(0) = \frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} \left(1+\frac{3i}{4\alpha}\right) + O(\alpha^{-\frac{5}{2}}),$$

$$h_1'(0) = -\frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}).$$

Hence, from (73),

$$\sigma_1 = \frac{2}{\alpha}\,\mathcal{I}\,\frac{\pi^{\frac{1}{2}}e^{-i\alpha-\frac{1}{4}\pi i}}{\alpha^{\frac{1}{2}}}\Bigg[-\frac{e^{-i\alpha}}{2\pi\alpha}\left\{\frac{1}{2}-\frac{7i}{32\alpha}+\frac{5i}{16\alpha}+\frac{2^{\frac{1}{2}}i}{8\alpha}\right\}$$

$$+\frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left\{1+\frac{3i}{4\alpha}+\frac{i}{\alpha}\left(\frac{13}{16}+\frac{3}{2}\right)\right\}$$

$$-\frac{e^{\frac{1}{4}\pi i+i\alpha}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left\{1+\frac{3i}{4\alpha}-\frac{i}{4\alpha}\left(1-\frac{9}{4}\right)\right\}\Bigg]+O(\alpha^{-\frac{9}{2}})$$

$$=\frac{2}{\alpha}\,\mathcal{I}\,\frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}}\,e^{-i\alpha-\frac{1}{4}\pi i}\Bigg[-\frac{e^{-i\alpha}}{4\pi\alpha}\left(1+\frac{3i}{16\alpha}\right.$$

$$\left.+\frac{2^{\frac{1}{2}}i}{4\alpha}\right)+\frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left(1+\frac{49i}{16\alpha}\right)-\frac{e^{\frac{1}{4}\pi i+i\alpha}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\left(1+\frac{17i}{16\alpha}\right)\Bigg]+O(\alpha^{-\frac{9}{2}}).$$

Similarly,

$$\sigma_2 = \frac{2}{\alpha}\,\mathcal{I}\left[-\frac{e^{-i\alpha}}{2\pi\alpha}h_2(1)+\frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}h_2(0)-\frac{e^{\frac{1}{4}\pi i+i\alpha}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}h_2(0)\right]+O(\alpha^{-\frac{9}{2}})$$

$$=\frac{2}{\alpha}\,\mathcal{I}\Bigg[-\frac{e^{-i\alpha}}{2\pi\alpha^2}\left\{h_{1,o}(1-2^{\frac{1}{2}})+\frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}h_1(1)\right\}$$

$$- \frac{e^{-\frac{1}{4}\pi i}}{4\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} h_{1,o} + \frac{e^{\frac{1}{4}\pi i + i\alpha}}{16\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} h_{1,o} \Bigg] + O(\alpha^{-\frac{9}{2}}) = \frac{2}{\alpha}\mathcal{J} \frac{\pi^{\frac{1}{2}} e^{-i\alpha - \frac{1}{4}\pi i}}{\alpha^{\frac{1}{2}}}$$

$$\Bigg[- \frac{e^{-i\alpha}}{2\pi\alpha^2} \Big\{ \frac{i}{8} (1 - 2^{\frac{1}{2}}) + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{32\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}} \Big\} - \frac{e^{\frac{1}{4}\pi i}}{32\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} + \frac{ie^{\frac{1}{4}\pi i + i\alpha}}{128\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} \Bigg]$$

$$+ O(\alpha^{-\frac{9}{2}}) .$$

In general $\sigma_n = O(\alpha^{-\frac{3}{2}-n})$ for $n \geq 1$ so that

(79)

$$\sigma = \sigma_o + \sigma_1 + \sigma_2 + O(\alpha^{-\frac{9}{2}})$$

$$= \frac{2}{\alpha}\mathcal{J} \Bigg[i\alpha + 1 - \frac{i}{4\alpha} + \frac{1}{4\alpha^2} + \frac{3i}{16\alpha^3} + \frac{ie^{-i\alpha}}{2\alpha^2} - \frac{3e^{-i\alpha}}{2\alpha^3}$$

$$- \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{4\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} \left(1 + \frac{3i}{16\alpha} + \frac{2^{\frac{1}{2}} i}{4\alpha}\right) - \frac{ie^{-i\alpha}}{2\alpha^2} \times$$

$$\left(1 + \frac{49i}{16\alpha}\right) - \frac{1}{8\alpha^2}\left(1 + \frac{17i}{16\alpha}\right) - \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{4\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} \Big\{ \frac{i}{4} (1 - 2^{\frac{1}{2}}) + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{1/2}\alpha^{1/2}} \Big\} - \frac{e^{-i\alpha}}{32\alpha^3} + \frac{i}{128\alpha^3} \Bigg]$$

$$+ O(\alpha^{-\frac{9}{2}}) = \frac{2}{\alpha}\mathcal{J} \Bigg[i\alpha + 1 - \frac{i}{4\alpha} + \frac{1}{8\alpha^2} + \frac{i}{16\alpha^3}$$

$$- \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{4\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} \left(1 + \frac{7i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) \Bigg] + O(\alpha^{-\frac{9}{2}})$$

$$= 2\left[1 - \frac{1}{4\alpha^2} + \frac{\cos(2\alpha - \frac{1}{4}\pi)}{4\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}} + \frac{7\sin(2\alpha - \frac{1}{4}\pi)}{64\pi^{\frac{1}{2}}\alpha^{\frac{7}{2}}} + \frac{1}{16\alpha^4}\right.$$

$$\left. - \frac{1}{64\pi\alpha^4}\sin(4\alpha - \tfrac{1}{2}\pi)\right] + O(\alpha^{-\frac{9}{2}}).$$

The first three terms are in agreement with the result obtained by Levine and Wu[16]; their calculations did not extend beyond these terms.

7. The sound-hard disc

For the sound-hard disc on which the normal derivative of the field vanishes, a general expression for the field is

$$u(\underline{R}) = u_0(r,z) + \frac{1}{4\pi}\int_S [u]\frac{\partial}{\partial z_1}\psi(\underline{R},\underline{R}_1)\,dS$$

where $[u] = u(r_1, +0) - u(r_1, -0)$. It has been shown in an earlier paper (Jones[4]) that the integral equation resulting from the imposition of the boundary condition $\partial u/\partial n = 0$ can be converted into

$$\int_x^1 g'(r_1)\frac{\cos\{\alpha(r_1^2 - x^2)^{1/2}\}}{(r_1^2 - x^2)^{1/2}}\,dr_1 + i\int_0^1 g(r_1)\frac{d}{dr_1}\frac{\sin\{\alpha(r_1^2 - x^2)^{1/2}\}}{(r_1^2 - x^2)^{1/2}}\,dr_1$$

$$= -\frac{2}{x}\int_0^x g^{(0)}(v)\frac{v\cosh\{\alpha(x^2 - v^2)^{1/2}\}}{(x^2 - v^2)^{1/2}}\,dv$$

where $g^{(o)}(v) = \left[\frac{\partial u_o(v,z)}{\partial z}\right]_{z=o}$ and $g(r_1) = [u]$. In the earlier paper we applied the inverse operator of the first term. In this case we integrate the second term by parts and apply the edge condition $g(1) = 0$. We arrive at

$$\int_x^1 g'(r_1)\,\frac{\cos\{\alpha(r_1^2 - x^2)^{1/2}\}}{(r_1^2 - x^2)^{1/2}}\,dr_1 - ig(0)\,\frac{\sin\alpha(-x^2)^{1/2}}{(-x^2)^{1/2}}$$

$$-i\int_o^1 g'(r_1)\,\frac{\sin\{\alpha(r_1^2 - x^2)^{1/2}\}}{(r_1^2 - x^2)^{1/2}}\,dr_1 = -\frac{2}{x}\int_o^x v\,g^{(o)}(v)\,\frac{\cosh\{\alpha(x^2 - v^2)^{1/2}\}}{(x^2 - v^2)^{1/2}}\,dv.$$

Make the change of variables $r_1 = t^{1/2}$, $x = z^{1/2}$, $v = y^{1/2}$. Then

$$\int_z^1 \frac{g'(t^{1/2})}{t^{\frac{1}{2}}}\;\frac{\cos\{\alpha(t-z)^{1/2}\}}{(t-z)^{1/2}}\,dt - 2ig(o)\,\frac{\sin\alpha\,(-z)^{1/2}}{(-z)^{1/2}}$$

$$-\,i\int_o^1 \frac{g'(t^{1/2})}{t^{1/2}}\;\frac{\sin\{\alpha(t-z)^{\frac{1}{2}}\}}{(t-z)^{1/2}}\,dt = -\frac{2}{z^{1/2}}\int_o^z g^{(0)}(t^{1/2})\;\frac{\cosh\{\alpha(z-t)^{1/2}\}}{(z-t)^{1/2}}\,dt$$

valid for $0 < z < 1$.

This is analogous to (6) but contains an extra term. Nevertheless, we attempt to follow the same procedure and write the equation as

$$\int_o^1 \frac{g'(t^{1/2})}{t^{1/2}}\;\frac{e^{-i\alpha(t-z)^{1/2}}}{(t-z)^{1/2}}\,dt - \int_o^z \frac{g'(t^{1/2})}{t^{1/2}}\;\frac{\cos\{\alpha(t-z)^{1/2}\}dt}{(t-z)^{1/2}}$$

$$-2ig(0)\,\frac{\sin\alpha(-z)^{1/2}}{(-z)^{1/2}} = -\frac{2}{z^{1/2}}\int_o^z g^{(o)}(t^{1/2})\times$$

$$\cdot\,\frac{\cosh\{\alpha(z-t)^{1/2}\}}{(z-t)^{1/2}}\,dt,$$

Multiply both sides by cos $\{\alpha(x-z)^{1/2}\}/(x-z)^{1/2}$ and integrate with respect to z from 0 to x. On account of (8), (9) and (B3) with $t = 0$ the equation becomes

$$\int_0^1 \frac{g'(t^{1/2})}{t^{1/2}} \left\{\int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{e^{-i\alpha y}}{y}\, dy - \pi i H(x-t)\right\}dt + \int_0^x \frac{g'(t^{\frac{1}{2}})}{t^{1/2}} \pi i \operatorname{sgn}(x-t)dt - 2ig(0)\times \int_{-x^{\frac{1}{2}}}^{x^{\frac{1}{2}}} \frac{\sin \alpha y}{y}\, dy = -2\int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}}{z^{1/2}(x-z)^{1/2}} \int_0^z g^{(o)}(t^{1/2}) \frac{\cosh \alpha(z-t)^{1/2}}{(z-t)^{1/2}}\, dt\, dz.$$

Instead of taking a derivative of this equation, as we did with (10), integrate the first term by parts with the consequence, since $g(1) = 0$, that

$$\int_0^1 \frac{g(t^{1/2})}{t^{1/2}} \left\{\frac{e^{-i\alpha(t^{1/2}+x^{1/2})}}{t^{1/2}+x^{1/2}} - \frac{e^{-i\alpha(t^{1/2}-x^{1/2})}}{t^{1/2}-x^{1/2}}\right\}dt = 2\int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}}{z^{1/2}(x-z)^{1/2}} \times \cdot \int_0^z g^{(o)}(t^{1/2}) \frac{\cosh \alpha(z-t)^{1/2}}{(z-t)^{1/2}}\, dt\, dz.$$

The change of variables $t = w^2$, $x = v^2$ now gives

$$(80) \qquad \int_0^1 g(w) \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw = G^{(o)}(v)$$

where

$$(81) \qquad G^{(o)}(v) = \int_0^{v^2} \frac{\cos\{\alpha(v^2-z)^{1/2}\}}{z^{1/2}(v^2-z)^{1/2}} \int_0^z g^{(o)}(t^{1/2}) \frac{\cosh \alpha(z-t)^{1/2}}{(z-t)^{1/2}}\, dt dz.$$

The integral equation (80) bears a strong resemblance to (11) but the kernels are in fact different. However, an integral equation of the second kind can be derived by similar means. Let

$$(82) \qquad G(v) = \int_0^1 g(w) \frac{e^{-i\alpha(w-v)}}{w-v}\, dw.$$

Then

$$g(w)e^{-i\alpha w} = -\frac{1}{\pi^2 w^{1/2}(1-w)^{1/2}}\int_0^1 \frac{v^{1/2}(1-v)^{1/2}}{v-w}\, G(v)e^{-i\alpha v}dv + \frac{C}{w^{1/2}(1-w)^{1/2}}.$$

Choose C so that $g(1) = 0$; then

$$(83)\quad g(w) = -\left(\frac{1-w}{w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi^2}\int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{G(v)e^{-i\alpha v}}{v-w}\, dv.$$

An alternative choice of C would be the one which makes g bounded at the origin. This leads to

$$(84)\quad g(w) = -\left(\frac{w}{1-w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi^2}\int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{G(v)e^{-i\alpha v}}{v-w}\, dv.$$

The difference between the right hand sides of (83) and (84) is

$$-\frac{e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \frac{G(v)e^{-i\alpha v}}{v^{1/2}(1-v)^{1/2}}\, dv$$

$$= -\frac{e^{i\alpha w}}{\pi^2 w^{1/2}(1-w)^{1/2}} \int_0^1 \frac{1}{v^{1/2}(1-v)^{1/2}} \times$$

$$\int_0^1 \frac{g(x)e^{-i\alpha x}}{x - v}\, dx\, dv$$

from (82). This quantity vanishes since

$$\int_0^1 \frac{dv}{v^{1/2}(1-v)^{1/2}(x-v)} = 0.$$

Hence the forumlae (83) and (84) are in agreement and either gives g(w) bounded and vanishing at the edge. Equation (83) is preferred because it ensures that g(1) = 0 even when approximation is used and because the behaviour near the edge is more important than that near the centre. Note that (84) corresponds to the formula (14) for wf(w). But it cannot be deduced that (83) would also be valid for wf(w) by the same reasoning because it is necessary that not only wf(w) but also f(w) be bounded at the origin and so (83) would fail in this case.

With g(w) given by (83)

$$\int_0^1 g(w)\,\frac{e^{-i\alpha(w+v)}}{w+v}\,dw = -\frac{1}{\pi^2}\int_0^1 \frac{e^{-i\alpha v}}{w+v}\left(\frac{1-w}{w}\right)^{1/2}\int_0^1\left(\frac{t}{1-t}\right)^{1/2} \times$$

$$\cdot\ \frac{G(t)e^{-i\alpha t}}{t-w}\ dt\ dw$$

$$= -\frac{e^{-i\alpha v}}{\pi^2}\int_0^1\left(\frac{t}{1-t}\right)^{1/2} G(t)e^{-i\alpha t}\ \frac{\pi}{t+v}\left(\frac{1+v}{v}\right)^{1/2} dt$$

from (E4). Hence

$$(85)\quad G(v) = -G^{(o)}(v) - \left(\frac{1+v}{v}\right)^{1/2} \frac{e^{-i\alpha v}}{\pi} \int_o^1 \left(\frac{t}{1-t}\right)^{1/2} \frac{G(t)e^{-i\alpha t}}{t+v}\, dt \quad (0<v<1)$$

which is completely analogous to (18).

8. The Plane Wave at Normal Incidence

For a plane wave at normal incidence $g^{(o)}(v) = -i\alpha$ and then, from (81),

$$G^{(o)}(v) = -2i \int_o^{v^2} \frac{\cos\{\alpha(v^2-z)^{1/2}\}}{z^{1/2}(v^2-z)^{1/2}} \sinh \alpha z^{1/2}\, dz$$

$$= -\ 4i\ \mathrm{Si}(\alpha v)$$

on a use of (B3). Put

$$(86)\qquad G(v) = -2\int_{-2v}^{\infty} \frac{e^{-i\alpha(t+v)}}{t+v}\, dt + G_1(v).$$

in (85). Then, because of (21),

$$G_1(v) = 2\int_o^{\infty} \frac{e^{-i\alpha(t+v)}}{t+v}\, dt - \left(\frac{1+v}{v}\right)^{1/2} \frac{e^{-i\alpha v}}{\pi} \int_o^1 \left(\frac{w}{1-w}\right)^{1/2} \frac{e^{-i\alpha w}}{w+v}\{G_1(w) - 2\int_o^{\infty} \frac{e^{-i\alpha(t-w)}}{t-w}\, dt\}dw.$$

Since (E14)

$$(87)\qquad \int_o^1 \left(\frac{w}{1-w}\right)^{1/2} \frac{dw}{(w+v)(w-t)} = \frac{\pi}{t+v}\left\{\left(\frac{v}{1+v}\right)^{1/2} - \left(\frac{t}{t-1}\right)^{1/2} H(t-1)\right\},$$

$$G_1(v) = 2\left(\frac{1+v}{v}\right)^{1/2} e^{-i\alpha v}\int_1^{\infty} \left(\frac{t}{t-1}\right)^{1/2} \frac{e^{-i\alpha t}}{t+v}\, dt - \left(\frac{1+v}{v}\right)^{1/2} \frac{e^{-i\alpha v}}{\pi}\int_o^1 \left(\frac{w}{1-w}\right)^{1/2} \frac{G_1(w)e^{-i\alpha w}}{w+v}\, dw.$$

Putting

$$G_1(v) = \left(\frac{1+v}{v}\right)^{1/2} e^{-i\alpha v} M(v) \tag{88}$$

we obtain

$$M(v) = \ell_1(v) - \frac{1}{\pi}\int_0^1 \left(\frac{1+w}{1-w}\right)^{1/2} \frac{M(w)e^{-2i\alpha w}}{w+v}\, dw \tag{89}$$

where

$$\begin{aligned} \ell_1(v) &= 2\int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{e^{-i\alpha t}}{t+v}\, dt \\ &= 2e^{-\frac{1}{4}\pi i-i\alpha} \int_0^\infty \frac{(1-it)^{1/2}e^{-\alpha t}}{t^{1/2}(1+v-it)}\, dt \end{aligned} \tag{90}$$

by deforming the contour into the straight line joining 1 to 1-i∞.

Equation (89) is analogous to (30). To find a solution valid at high frequencies we try

$$M(v) = \left(\frac{v}{1+v}\right)^{1/2} \int_0^\infty \frac{L_1(t)}{t-v} e^{-2i\alpha(t-v)}\, dt + M_2(v).$$

Then

$$\begin{aligned} M_2(v) = {} & \ell_1(v) - \left(\frac{v}{1+v}\right)^{1/2}\int_0^\infty \frac{L_1(t)e^{-2i\alpha(t-v)}}{t-v}\, dt + \left(\frac{v}{1+v}\right)^{1/2} \times \\ & \cdot \int_0^\infty \frac{L_1(t)e^{-2i\alpha t}}{t+v}\, dt - \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{L_1(t)e^{-2i\alpha t}}{t+v}\, dt \\ & - \frac{1}{\pi}\int_0^1 \left(\frac{1+w}{1-w}\right)^{1/2} \frac{M_2(w)e^{-2i\alpha w}}{w+v}\, dw \end{aligned}$$

where advantage has been taken of (87). Choose L_1 so that

$$\int_0^\infty L_1(t)\left\{\frac{e^{-2i\alpha(t-v)}}{t-v}-\frac{e^{-2i\alpha t}}{t+v}\right\}dt=\left(\frac{1+v}{v}\right)^{1/2}\ell_1(v).$$

This integral equation is studied in Appendix D (D17) and its solution (D24) is

$$L_1(y)=\frac{\alpha e^{i\alpha y}}{4\pi i}\int_0^\infty\left(\frac{1+x}{x}\right)^{1/2}\ell_1(x)e^{-i\alpha x}M_2(x,y)\,dx$$

where

$$(91)\quad M_2(x,y)=\frac{4xyH_0^{(2)}(\alpha x)J_1(\alpha y)-4x^2H_1^{(2)}(\alpha x)\,J_0(\alpha y)}{x^2-y^2}.$$

Hence in this case we form the iteration scheme

$$(92)\quad M_n(v)=\left(\frac{v}{1+v}\right)^{1/2}\int_0^\infty\frac{L_n(t)e^{-2i\alpha(t-v)}}{t-v}\,dt,$$

$$(93)\quad L_n(y)=\frac{\alpha e^{i\alpha y}}{4\pi i}\int_0^\infty\left(\frac{1+x}{x}\right)^{1/2}\ell_n(x)e^{-i\alpha x}M_2(x,y)\,dx\quad(n\geq 1),$$

$$(94)\quad \ell_n(v)=-\int_1^\infty\left(\frac{t}{t-1}\right)^{1/2}\frac{L_{n-1}(t)e^{-2i\alpha t}}{t+v}\,dt\quad(n\geq 2)$$

and ℓ_1 is given by (90).

It may be shown in a similar way to that of Section 4 that

$$(95)\quad L_n(y)=\frac{i}{\pi}\left(\frac{1+y}{y}\right)^{1/2}\ell_n(y)-\frac{2\alpha}{\pi^2}e^{i\alpha y+\frac{1}{4}\pi i}\int_0^\infty\frac{(1-ix)^{1/2}}{x^{1/2}}\ell_n(-ix)e^{-\alpha x}\times\left\{\frac{x^2K_1(\alpha x)J_0(\alpha y)-xyK_0(\alpha x)J_1(\alpha y)}{x^2+y^2}\right\}dx.$$

From this it can be deduced that, if αy is not small,

$$(96)\quad L_n(y) \sim \frac{i}{\pi}\left(\frac{1+y}{y}\right)^{1/2} \ell_n(y) + \frac{\ell_n(0)\, e^{i\alpha y+\frac{1}{4}\pi i} J_1(\alpha y)}{(32\pi\alpha)^{1/2}\, y} .$$

9. The Distribution on the Sound-hard Disc

Let the contribution of $G(v)-G_1(v)$ to g be g_0 and the contribution of M_n be g_n. Then, from (83) and (86),

$$(97)\quad g_0(w) = 2\left(\frac{1-w}{w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi^2} \int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{e^{-i\alpha v}}{v-w} \int_0^\infty \frac{e^{-i\alpha(t-v)}}{t-v}\, dt\, dv$$

and from (88)

$$(98)\quad g_n(w) = -\left(\frac{1-w}{w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi^2} \int_0^1 \left(\frac{1+v}{1-v}\right)^{1/2} \frac{Mn(v)\, e^{-2i\alpha v}}{v-w}\, dv.$$

Now, for $0 < w < 1$ and $t > 0$, (E12)

$$(99)\quad \int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \pi^2\left(\frac{w}{1-w}\right)^{1/2} \delta(t-w) + \pi\left(\frac{t}{t-1}\right)^{1/2} \frac{H(t-1)}{w-t}.$$

Therefore

$$(100)\quad g_0(w) = -2 + \frac{2}{\pi}\left(\frac{1-w}{w}\right)^{1/2} e^{i\alpha w} \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{e^{-i\alpha t}}{t-w}\, dt.$$

Also

$$g_n(w) = -\left(\frac{1-w}{w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi^2} \int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{1}{v-w} \int_0^\infty \frac{L_n(t)e^{-2i\alpha t}}{t-v} \, dt \, dv$$

$$= L_n(w)e^{-i\alpha w} - \left(\frac{1-w}{w}\right)^{1/2} \frac{e^{i\alpha w}}{\pi} \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \times \frac{L_n(t)}{t-w} e^{-2i\alpha t} dt. \tag{101}$$

We shall not attempt to take these forumlae further although approximations for large α on the lines of Section 5 are possible. Instead we shall concentrate on the scattering coefficient.

10. The Scattering Coefficient of the Sound-hard Disc

The scattering coefficient of the sound-hard disc is $-2\mathcal{R}\int_0^1 g(w)\, w dw$. Denote by σ_n the contribution of g_n. Then, from (100),

$$\sigma_o = -2\mathcal{R}\int_0^1 \{-2w + \frac{2}{\pi} w^{1/2}(1-w)^{1/2} e^{i\alpha w} \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{e^{-i\alpha t}}{t-w} dt\} \, dw$$

$$\sim 2\mathcal{R}\left[1 - \frac{2}{\pi^{1/2}} \int_0^1 w^{1/2}(1-w)^{1/2} e^{\frac{1}{4}\pi i} \int_\alpha^\infty \frac{e^{-i(1-w)x}}{x^{1/2}} \left(1 - \frac{i}{4x} + \frac{3}{32x^2}\right) dx \, dw\right].$$

Now, for $x \geq \alpha$,

$$\int_0^1 w^{1/2}(1-w)^{1/2}\, e^{iwx}\, dw \sim \frac{\pi^{1/2} e^{ix-\frac{3}{4}\pi i}}{2x^{3/2}} \{1 + \frac{3i}{4x} + \frac{15}{32x^2}\} + \frac{\pi^{\frac{1}{2}} e^{\frac{3}{4}\pi i}}{2x^{3/2}} (1 - \frac{3i}{4x}).$$

Hence

(102)

$$\sigma_o \sim 2\mathcal{R}\left[1 - \int_\alpha^\infty \frac{e^{-ix}}{x^2} (1 - \frac{i}{4x} + \frac{3}{32x^2}) \{e^{ix-\frac{1}{2}\pi i}(1 + \frac{3i}{4x} + \frac{15}{32x^2}) + e^{\pi i}(1- \frac{3i}{4x})\}\, dx\right]$$

$$\sim 2\mathcal{R}\left[1 - \int_\alpha^\infty \{\frac{-i}{x^2} (1 + \frac{i}{2x} + \frac{3}{8x^2}) - \frac{e^{-ix}}{x^2}(1 - \frac{i}{x})\}\, dx\right]$$

$$\sim 2\mathcal{R}\left[1 + i(\frac{1}{\alpha} + \frac{i}{4\alpha^2} + \frac{1}{8\alpha^3}) - \frac{ie^{-i\alpha}}{\alpha^2} + \frac{e^{-i\alpha}}{\alpha^3}\right]$$

$$\sim 2\left[1 - \frac{1}{4\alpha^2} - \frac{1}{\alpha^2} \sin \alpha + \frac{1}{\alpha^3} \cos \alpha\right].$$

Turning now to $\sigma_n (n \geq 1)$ we have

(103)

$$\sigma_n = 2\mathcal{R}\left[- \int_0^1 \{wL_n(w)e^{-i\alpha w}\, dw - w^{1/2}(1-w)^{1/2}\, \frac{e^{i\alpha w}}{\pi} \int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{L_n(t)}{t-w} e^{-2i\alpha t}\, dt\}dw\right].$$

Now

(104)

$$- \int_0^1 wL_n(w)e^{-i\alpha w}\, dw = \int_1^\infty wL_n(w)e^{-i\alpha w}\, dw - \int_0^\infty wL_n(w)e^{-i\alpha w}\, dw.$$

Also

$$\int_0^\infty wL_n(w)\, e^{-i\alpha w}\, dw = \frac{\alpha}{4\pi i}\int_0^\infty w \int_0^\infty \left(\frac{1+x}{x}\right)^{1/2} \ell_n(x) e^{-i\alpha x}\, M_2(x,w)\, dx\, dw.$$

Since, from (63), $\int_0^\infty \frac{w\, J_0(\alpha w)}{x^2 - w^2}\, dw = \frac{1}{2}\pi Y_0(\alpha x)$ and $\int_0^\infty \frac{w^2 J_1(\alpha w)}{x^2 - w^2}\, dw = \frac{1}{2}\pi Y_1(\alpha x)$

the right handside becomes, after a use of the Wronskian relation,

$$-\frac{1}{\pi i}\int_0^\infty x^{1/2}(1+x)^{1/2}\, \ell_n(x) e^{-i\alpha x}\, dx$$

$$\sim -\frac{e^{-\frac{3}{4}\pi i}\, \ell_n(0)}{2\pi^{1/2} i\, \alpha^{3/2}}.$$

In the first integral of (104) the form (96) can be used and, furthermore, the asymptotic expression $(2/\pi\alpha y)^{1/2}\cos(\alpha y - \frac{3}{4}\pi)$ for $J_1(\alpha y)$ can be employed. Hence

$$\int_1^\infty wL_n(w) e^{-i\alpha w}\, dw \sim \int_1^\infty \left\{\frac{i}{\pi}\, w^{1/2}(1+w)^{1/2}\, \ell_n(w) e^{-i\alpha w} + \frac{\ell_n(0)\, e^{\frac{1}{4}\pi i}}{4\pi\alpha w^{1/2}}\cos(\alpha w - \tfrac{3}{4}\pi)\right\} dw$$

$$\sim \frac{2^{\frac{1}{2}} i}{\pi\alpha}\, e^{-\frac{1}{2}\pi i - i\alpha}\left[\ell_n(1) - \frac{i}{\alpha}\left\{\tfrac{3}{4}\ell_n(1) + \ell_n'(1)\right\}\right] - \frac{\ell_n(0) e^{\frac{1}{4}\pi i}}{4\pi\alpha^2}\sin(\alpha - \tfrac{3}{4}\pi).$$

Consequently

(105)

$$-\int_0^1 wL_n(w) e^{-i\alpha w} dw \sim \frac{2^{1/2} e^{-i\alpha}}{\pi\alpha}\left[\ell_n(1) - \frac{i}{\alpha}\left\{\tfrac{3}{4}\ell_n(1) + \ell_n'(1)\right\}\right] - \frac{\ell_n(0)\, e^{\frac{1}{4}\pi i}}{4\pi\alpha^2}\sin(\alpha - \tfrac{3}{4}\pi)$$

$$- \frac{e^{-\frac{1}{4}\pi i}\, \ell_n(0)}{2\pi^{1/2}\alpha^{3/2}}.$$

In the second integral of (103) the form (96) can be used with J_1 replaced by the first term of its asymptotic expansion. We thus obtain

(106)

$$\frac{1}{\pi}\int_0^1 w^{1/2}(1-w)^{1/2}e^{i\alpha w}\int_1^\infty \left(\frac{t}{t-1}\right)^{1/2}\frac{e^{-2i\alpha t}}{t-w}\left\{\frac{i}{\pi}\left(\frac{t+1}{t}\right)^{1/2}\ell_n(t)\,-\right.$$

$$\left.-\,\frac{\ell_n(0)}{8\pi\alpha t^{3/2}}\left(1+ie^{2i\alpha t}\right)\right\}dt\,dw.$$

Consider firstly the terms inside the brace { } which do not involve $e^{2i\alpha t}$. Now

$$\int_0^1 \frac{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}{t-w}\,e^{i\alpha w}dw = e^{\frac{3}{4}\pi i}\int_0^\infty \frac{x^{\frac{1}{2}}(1-ix)^{1/2}e^{-\alpha x}}{t-ix}\,dx - e^{\frac{1}{4}\pi i}\int_0^\infty \frac{x^{\frac{1}{2}}(1+ix)^{\frac{1}{2}}\,e^{i\alpha-\alpha x}}{t-1-ix}\,dx$$

$$\sim ie^{\frac{3}{4}\pi i+i\alpha t}\int_\alpha^\infty \frac{e^{-itx}}{x^{3/2}}\left(\tfrac{1}{2}! - \frac{i}{2x}\,\tfrac{3}{2}!\right)dx$$

$$-\,ie^{\frac{1}{4}\pi i+i\alpha t}\int_\alpha^\infty \frac{e^{-i(t-1)x}}{x^{3/2}}\left(\tfrac{1}{2}! + \frac{i}{2x}\cdot\tfrac{3}{2}!\right)dx$$

from (52). Hence the terms in the { } of (106) not involving $e^{2i\alpha t}$ have asymptotic behaviour

$$\frac{ie^{\frac{3}{4}\pi i}}{2\pi^{1/2}}\int_\alpha^\infty \frac{1}{x^{3/2}}\left\{1+ie^{ix} - \frac{3i}{4x}(1-ie^{ix})\right\}\int_1^\infty \frac{e^{-it(\alpha+x)}}{(t-1)^{1/2}}\left\{\frac{i}{\pi}(t+1)^{1/2}\ell_n(t) - \frac{\ell_n(0)}{8\pi\alpha t}\right\}dt\,dx$$

$$\sim \frac{-e^{-i\alpha}}{2}\int_\alpha^\infty \frac{e^{-ix}}{x^{3/2}}\left\{1+ie^{ix} - \frac{3i}{4x}(1-ie^{ix})\right\}\left[\left\{\frac{i}{\pi}\,2^{1/2}\,\ell_n(1) - \frac{\ell_n(0)}{8\pi\alpha}\right\}\frac{1}{(x+\alpha)^{1/2}}\right.$$

$$\left.+\,\frac{2^{1/2}}{2(x+\alpha)^{3/2}\pi}\left\{\frac{1}{4}\,\ell_n(1) + \ell_n'(1)\right\}\right]dx.$$

The substitution $x = \alpha \tan^2 \theta$ shows that

$$\int_\alpha^\infty \frac{dx}{(x+\alpha)^{1/2} x^{3/2}} = \frac{2}{\alpha}(2^{1/2} - 1),$$

$$\int_\alpha^\infty \frac{dx}{(x+\alpha)^{1/2} x^{5/2}} = \frac{2}{3\alpha^2}(2 - 2^{1/2}),$$

$$\int_\alpha^\infty \frac{dx}{(x+\alpha)^{3/2} x^{3/2}} = \frac{2}{\alpha^2}\left(\frac{3}{2^{1/2}} - 2\right).$$

Hence the value of the integral is

(107)
$$-\frac{e^{-i\alpha}}{\pi\alpha}\left[(2^{1/2} - 2)\,\ell_n(1) + \frac{i\ell_n(1)}{\alpha}\left(\frac{7}{8} - \frac{3}{2^{3/2}}\right) + \frac{i\ell_n'(1)}{\alpha}\left(\frac{3}{2} - 2^{1/2}\right) - \frac{i\ell_n(0)}{8\alpha}(2^{1/2} - 1) + \frac{\ell_n(1)e^{-i\alpha}}{2\alpha}\right].$$

The remaining term of (106) is

$$(108) \quad -\frac{i\ell_n(0)}{8\pi^2\alpha}\int_0^1 w^{1/2}(1-w)^{1/2} e^{i\alpha w}\int_1^\infty \frac{dt}{t(t-1)^{1/2}(t-w)}\,dw.$$

The inner integral is a function of w with no singularities in (0,1) except near $w = 1$ where it equals $(1-w)^{-1/2}\int_c^\infty \frac{dt}{t^{1/2}(t+1)} + O(1) = \pi(1-w)^{-\frac{1}{2}} + O(1)$. Hence the first term in the asymptotic expansion of (108) is

$$(109) \qquad -\frac{\ell_n(0)e^{i\alpha}}{8\pi\alpha^2}.$$

Combining (105),(107) and (109) we have

$$\text{(110)}\quad \sigma_n \sim 2\Re\left[\frac{e^{-i\alpha}}{\pi\alpha}\left\{\left(2-\frac{7i}{8\alpha}-\frac{e^{-i\alpha}}{2\alpha}\right)\ell_n(1)-\frac{3i}{2\alpha}\ell_n'(1)+\frac{i2^{1/2}\ell_n(0)}{8\alpha}\right\}-\frac{e^{-\frac{1}{4}\pi i}}{2\pi^{1/2}\alpha^{3/2}}\ell_n(0)\right].$$

The error in this expression is $O(\ell_n/\alpha^{5/2})$.

It is also convenient to have available asymptotic formulae for ℓ_n. From (94) and (96), for $n \geq 2$,

$$\ell_n(v) \sim -\int_1^\infty \left(\frac{t}{t-1}\right)^{1/2} \frac{e^{-2i\alpha t}}{t+v}\left\{\frac{i}{\pi}\left(\frac{1+t}{t}\right)^{1/2}\ell_{n-1}(t)-\frac{\ell_{n-1}(0)}{8\pi\alpha t^{3/2}}\left(1+ie^{2i\alpha t}\right)\right\}dt.$$

It is sufficient for our purposes to observe that this implies that

$$\ell_n(v) \sim -\frac{i}{\pi}e^{-2i\alpha-\frac{1}{4}\pi i}\frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}\ell_{n-1}(1)}{(2\alpha)^{1/2}(1+v)}+\frac{i\ell_{n-1}(0)}{8\pi\alpha}\int_1^\infty \frac{dt}{t(t-1)^{1/2}(t+v)}$$

for $v \geq 0$, the error being $O(\ell_{n-1}/\alpha^{3/2})$. The substitution $t = 1 + x^2$ shows that

$$\int_1^\infty \frac{dt}{t(t-1)^{1/2}(t+1)} = \pi\left(1-\frac{1}{2^{1/2}}\right), \text{ that } \int_1^\infty \frac{dt}{t^2(t-1)^{1/2}} = \frac{1}{2}\pi \text{ and that}$$

$$\int_1^\infty \frac{dt}{t(t-1)^{1/2}(t+1)^2} = \pi\left(1-\frac{5}{2^{5/2}}\right).$$ Hence

$$\text{(111)}\quad \ell_n(1) \sim -\frac{e^{\frac{1}{4}\pi i-2i\alpha}}{2\pi^{1/2}\alpha^{1/2}}\ell_{n-1}(1)+\frac{i\ell_{n-1}(0)}{8\alpha}\left(1-\frac{1}{2^{1/2}}\right),$$

$$\text{(112)}\quad \ell_n'(1) \sim \frac{e^{\frac{1}{4}\pi i-2i\alpha}}{4\pi^{1/2}\alpha^{1/2}}\ell_{n-1}(1)-\frac{i\ell_{n-1}(0)}{8\alpha}\left(1-\frac{5}{2^{5/2}}\right),$$

(113) $\ell_n(0) \sim -\dfrac{e^{\frac{1}{4}\pi i-2i\alpha}}{\pi^{1/2}\alpha^{1/2}} \quad \ell_{n-1}(1) + \dfrac{i\ell_{n-1}(0)}{16\alpha}$.

The corresponding formulae for ℓ_1 are deduced directly from (90) and

$$\ell_1(v) \sim \frac{2\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi i-i\alpha}}{\alpha^{1/2}(1+v)} \{1 + \frac{i(1-v)}{4\alpha(1+v)}\}$$

so that

$$\ell_1(1) = \frac{\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi i-i\alpha}}{\alpha^{1/2}} \{1+O(\alpha^{-2})\},$$

$$\ell_1'(1) = -\frac{\pi^{1/2}e^{-\frac{1}{4}\pi i-i\alpha}}{2\alpha^{1/2}} \{1+O(\alpha^{-1})\},$$

$$\ell_1(0) = \frac{2\pi^{1/2}e^{-\frac{1}{4}\pi i-i\alpha}}{\alpha^{1/2}} \{1 + O(\alpha^{-1})\}.$$

Hence, from (110),

$$\sigma_1 = 2\mathcal{R}\,\frac{\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi i-i\alpha}}{\alpha^{3/2}} \left[\frac{e^{-i\alpha}}{\pi}\{2-\frac{i}{8\alpha} - \frac{e^{-i\alpha}}{2\alpha} + \frac{2^{\frac{1}{2}}i}{4\alpha}\} - \frac{e^{-\frac{1}{4}\pi i}}{\pi^{1/2}\alpha^{1/2}}\right] + O(\alpha^{-3}),$$

$$\sigma_2 = 2\mathcal{R}\left[\frac{2e^{-i\alpha}}{\pi\alpha}\ \ell_2(1) - \frac{e^{-\frac{1}{4}\pi i}}{2\pi^{1/2}\alpha^{3/2}}\ \ell_2(0)\right] + O(\alpha^{-3})$$

$$= 2\mathcal{R}\left[\frac{2e^{-i\alpha}}{\pi\alpha}\{-\frac{e^{\frac{1}{4}\pi i-2i\alpha}\ell_1(1)}{2\pi^{1/2}\alpha^{1/2}} + \frac{i\ell_1(0)}{8\alpha}(1-\frac{1}{2^{1/2}})\} + \frac{e^{-2i\alpha}\ell_1(1)}{2\pi^{1/2}\alpha^{3/2}\pi^{1/2}\alpha^{1/2}}\right] + O(\alpha^{-3})$$

$$= 2\mathcal{R}\,\frac{\pi^{\frac{1}{2}}e^{-\frac{1}{4}\pi i-i\alpha}}{\alpha^{2}}\left[-\frac{e^{\frac{1}{4}\pi i-3i\alpha}}{\pi^{3/2}} + \frac{ie^{-i\alpha}}{2\pi\alpha^{1/2}}(1-\frac{1}{2^{1/2}}) + \frac{e^{-2i\alpha}}{2\pi\alpha^{1/2}}\right] + O(\alpha^{-3})$$

from (111) and (113). Similarly

$$\sigma_3 = 2\mathcal{R}\,\frac{2e^{-i\alpha}}{\pi\alpha}\,\ell_3(1) + O(\alpha^{-3})$$

$$= 2\mathcal{R} - \frac{e^{\frac{1}{4}\pi i-2i\alpha}}{2\pi^{1/2}\alpha^{1/2}} \cdot \frac{2e^{-i\alpha}}{\pi\alpha}\,\ell_2(1) + O(\alpha^{-3})$$

$$= 2\mathcal{R}\,\frac{e^{\frac{1}{4}\pi i-6i\alpha}}{2\pi^{3/2}\alpha^{5/2}} + O(\alpha^{-3}).$$

In general $\sigma_n = O(\alpha^{-\frac{1}{2}n-1})$ so that

(114) $$\sigma = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + O(\alpha^{-3})$$

$$= 2\mathcal{R}\left[1 + \frac{i}{\alpha} - \frac{1}{4\alpha^2} + \frac{ie^{\frac{1}{2}\pi i-4i\alpha}}{\pi\alpha^2} - \frac{2i}{\pi^{1/2}\alpha^{3/2}}\,e^{\frac{1}{4}\pi i-2i\alpha} + \frac{3e^{\frac{1}{4}\pi i-2i\alpha}}{8\pi^{1/2}\alpha^{5/2}} - \frac{ie^{\frac{3}{4}\pi i-6i\alpha}}{2\pi^{3/2}\alpha^{5/2}}\right] + O(\alpha^{-3})$$

$$= 2\left[1 - \frac{2}{\pi^{1/2}\alpha^{3/2}}\sin(2\alpha-\tfrac{1}{4}\pi) + \frac{1}{\pi\alpha^2}\sin(4\alpha-\tfrac{1}{2}\pi) - \frac{1}{4\alpha^2} + \frac{3}{8\pi^{1/2}\alpha^{5/2}}\cos(2\alpha-\tfrac{1}{4}\pi) - \frac{1}{2\pi^{3/2}\alpha^{5/2}}\sin(6\alpha-\tfrac{3}{4}\pi)\right] + O(\alpha^{-3}) \quad .$$

This formula is in agreement with the result obtained by Levine and Wu[16] except that their factor of the term $\alpha^{-5/2}\cos(2\alpha - \frac{1}{4}\pi)$ is $\frac{1}{4}$ instead of $\frac{3}{8}$. The cause of the discrepancy has not been traced but is very likely due to an arithmetical error. Dr. Levine does not think it possible that the next stage of the approximation adopted by Levine and Wu would produce a term of comparable order.

11. Uniqueness

In this section we show that the solution of the integral equations (18) is unique. This immediately implies that the solution of (30) is unique. Furthermore, it has been already shown that (11) implies (18) i.e. a solution of (11) provides a solution of (18). On the other hand, if F is any bounded solution of (18) it is easy to verify that f defined by (14) satisfies (11). Therefore, there is an equivalence between the two integral equation for the types of solution under consideration. Hence a unique solution of (18) implies a unique solution of (11). But since (1) possesses a unique solution (Jones [27]) which is a solution of (11) it follows that our process provides the unique solution sought.

We turn now to the proof that the solution of (18) is unique. Since $f(w)$ is continuous in $0 \leq w < 1$, (13) shows that $F(v)$ is bounded and continuous in $0 < v < 1$. Near $v = 0$

$$F(v) = f(0)\int_0^{\epsilon} \frac{w\,dw}{w-v} + O(1)$$

where $1 >> \epsilon >> v$. Hence

$$F(v) = f(0)v\ \ell n\ (\epsilon/v) + O(1)$$

so that F is bounded at $v = 0$. Near $w = 1$, $f(w) = A(1-w)^{-\frac{1}{2}} + O((1-w)^{1/2})$ and so

$$F(v) = A\int_{1-\epsilon}^{1} \frac{(1-w)^{-\frac{1}{2}}}{w-v}dw + O(1) = O(1)$$

since $\int_0^1 \frac{w^{\frac{1}{2}}\,dw}{(1-w)^{1/2}(w-v)} = \pi$, which differs from the given integral by

O(1). Consequently a solution of (18) is required in which F(v) is continuous in $0 \leq v \leq 1$.

Suppose there are two such solutions of (18). Then their difference, $F_o(v)$, is of the same type and satisfies

(115)
$$F_o(v) = - \frac{v^{\frac{1}{2}} e^{-i\alpha v}}{\pi(1+v)^{1/2}} \int_o^1 \frac{(1-w)^{\frac{1}{2}}}{w^{1/2}(w+v)} F_o(w) e^{-i\alpha w} \, dw.$$

for v in (0,1). It follows that

$$\left| F_o(v) \right| \leq \frac{v^{\frac{1}{2}}}{\pi(1+v)^{1/2}} \max|F_o| \int_o^1 \frac{(1-w)^{\frac{1}{2}}}{w^{1/2}(w+v)} \, dw$$

$$\leq \{1 - \frac{v^{1/2}}{(1+v)^{1/2}}\} \max |F_o|.$$

If $\max |F_o(v)|$ occurs at $v = v_o$ we have

$$\max |F_o| \leq \{1 - \frac{v_o^{1/2}}{(1+v_o)^{1/2}}\} \max|F_o| .$$

If $\max |F_o| = 0$ there is nothing to prove and if $\max |F_o| \neq 0$ we must have $v_o = 0$. Now as $v \to 0$

$$F_o(v) = - \frac{v^{\frac{1}{2}}}{\pi} \int_o^\epsilon \frac{F_o(w) dw}{w^{1/2}(w+v)} + o(1)$$

$$= - \frac{v^{1/2}}{\pi} F_o(\xi) \int_o^\epsilon \frac{dw}{w^{1/2}(w+v)} + o(1)$$

where $0 \le \xi \le \epsilon$, by the mean value theorem. Since

$$\int_0^\epsilon \frac{dw}{w^{1/2}(w+v)} = \frac{2}{v^{1/2}} \tan^{-1} (\epsilon/v)^{1/2} = \frac{\pi}{v^{1/2}} + O(1),$$

(116) $F_o(v) + F_o(\xi) = o(1)$

from which may be deduced that $F_o(0) = 0$. Therefore $\max |F_o| = 0$ and $F_o \equiv 0$ i.e. the solution of (18) is unique.

A similar method may be employed to determine the error caused by stopping our iteration process at some n. Let $F_n(v) = v^{1/2} e^{-i\alpha v} I_n(v)/(1+v)^{1/2}$. Then

$$F_n(v) = \frac{v^{\frac{1}{2}} h_n(v) e^{-i\alpha v}}{(1+v)^{1/2}} - \frac{v^{\frac{1}{2}} e^{-i\alpha v}}{\pi(1+v)^{1/2}} \int_0^1 \frac{(1-w)^{\frac{1}{2}} F_n(w) e^{-i\alpha w}}{w^{1/2}(w+v)} dw.$$

As above

$$|F_n(v)| \le 2^{-\frac{1}{2}} |h_n(v)|\, v^{\frac{1}{2}} + \{1 - v^{\frac{1}{2}}/(1+v)^{\frac{1}{2}}\} \max|F_n|.$$

If $|F_n|$ has its maximum at $v = v_n \neq 0$,

$$\max|F_n| \le 2^{-\frac{1}{2}} |h_n(v_n)| (1+v_n)^{\frac{1}{2}} = O(\alpha^{-n+\frac{1}{2}})$$

from (76), (74) and (78). On the other hand if $|F_n|$ has its maximum at $v = 0$, then, as $v \to 0$, $F_n(v) + F_n(\xi) = o(1)$ by an argument similar to that leading to (116) and $F_n \equiv 0$. In either case we can say that the error in stopping the calculation at I_{n-1} is $O(\alpha^{-n+\frac{1}{2}})$ in F.

The integral equation (85) requires a more elaborate treatment and only the proof of uniqueness for $|\alpha| \leq \pi$ will be given here. Firstly the possibility that G has a logarithmic singularity at the origin cannot be excluded. The difference between two possible solutions G_o satisfies

$$(117)\quad G_o(v) = -\left(\frac{1+v}{v}\right)^{1/2} \frac{e^{-i\alpha v}}{\pi} \int_o^1 \left(\frac{t}{1-t}\right)^{1/2} \frac{G_o(t)e^{-i\alpha t}}{t+v}\, dt,$$

which may be written as

$$(118)\qquad G_o(v) = -\left(\frac{1+v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{\pi} \int_o^1 \left(\frac{t}{1-t}\right)^{\frac{1}{2}} G_o(t)e^{-i\alpha t}\left\{\frac{1}{t} - \frac{v}{t(t+v)}\right\} dt.$$

The form of the solution needed is $G_o(v) = A \ln v + \gamma(v)$ where $\gamma(v)$ is bounded and continuous in $0 \leq v \leq 1$. To examine whether such a solution of (118) is possible determine the behaviour of the right hand side of (118) as $v \rightarrow 0$. Now

$$\int_o^1 \frac{vG_o(t)e^{-i\alpha t}}{t^{1/2}(1-t)^{1/2}(t+v)}\, dt = \int_o^\epsilon \frac{vG_o(t)e^{-i\alpha t}}{t^{1/2}(1-t)^{1/2}(t+v)}\, dt + O(v).$$

In the integral over $(0,\epsilon)$ use the assumed form of G_o. Note that

$$\int_0^{\epsilon} \frac{v\gamma_o(t)e^{-i\alpha t}}{t^{1/2}(1-t)^{1/2}(t+v)}\,dt = \frac{\gamma_o(\xi)e^{-i\alpha\xi}}{(1-\xi)^{1/2}} \quad v\int_0^{\epsilon} \frac{dt}{t^{1/2}(t+v)}$$

$$= \frac{2\gamma_o(\xi)e^{-i\alpha\xi}}{(1-\xi)^{1/2}}\, v^{1/2} \tan^{-1} \sqrt{\frac{\epsilon}{v}}$$

$$= O(v^{\frac{1}{2}}).$$

Further

$$\int_0^{\epsilon} \frac{vt^{n-\frac{1}{2}}\ell n t}{t+v}\,dt = v^{n+\frac{1}{2}}\int_0^{\epsilon/v} \frac{t^{n-\frac{1}{2}}\ell n(vt)}{t+1}\,dt$$

$$= v^{n+\frac{1}{2}}\int_0^{\epsilon/v} \{t^{n-\frac{3}{2}} - t^{n-\frac{5}{2}} + \dots + (-)^{n-1}t^{-\frac{1}{2}} + \frac{(-)^n t^{-\frac{1}{2}}}{t+1}\}\, \ell n(vt)dt$$

$$= 2(-)^n v^{n+\frac{1}{2}} \int_0^{(\epsilon/v)^{\frac{1}{2}}} \frac{\ell n\, v + 2\ell n t}{t^2+1}\,dt + O(v)$$

$$= (-)^n \pi v^{n+\frac{1}{2}} \ell n\, v + O(v^{n+1}\, \ell n\, v) + O(v)$$

since $\int_0^\infty \frac{\ln t}{t^2+1}\, dt = 0$. Hence

$$(119) \qquad \int_0^1 \frac{vG_o(t)e^{-i\alpha t}dt}{t^{1/2}(1-t)^{1/2}(t+v)} = \pi A v^{1/2} \ln v + O(v^{\frac{1}{2}})$$

and the $O(v^{1/2})$ term is actually $\pi\gamma o(0)v^{\frac{1}{2}} + O(v\ln v)$. Consequently

$$G_o(v) = A \ln v - \frac{1}{\pi v^{1/2}} \int_0^1 \frac{G_o(t)e^{-i\alpha t}}{t^{1/2}(1-t)^{1/2}}\, dt + O(1).$$

One concludes that $G_o(v)$ can satisfy the given conditions only if

$$(120) \qquad \int_0^1 \frac{G_o(t)e^{-i\alpha t}}{t^{1/2}(1-t)^{1/2}}\, dt = 0$$

and it then follows from (118) that G_o also satisfies

$$(121) \qquad G_o(v) = \frac{v^{\frac{1}{2}}}{\pi} (1+v)^{\frac{1}{2}} e^{-i\alpha v} \int_0^1 \frac{G_o(t)e^{-i\alpha t}}{(1-t)^{1/2}t^{1/2}(t+v)}\, dt.$$

Since (121) can be rewritten as

$$G_o(v) = \frac{v^{\frac{1}{2}}}{\pi}(1+v)^{\frac{1}{2}}e^{-i\alpha v} \int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{G_o(t)e^{-i\alpha t}}{1+v} \left\{ \frac{1}{1-t} + \frac{1}{t+v} \right\} dt$$

it follows from (120) that G_o must also satisfy

$$(122) \qquad G_o(v) = \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{\pi} \int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{G_o(t)e^{-i\alpha t}}{t+v}\, dt.$$

The facts that G_o must comply with the condition (120) and also be a solution of (122) are consistent with the discussion in § 7 of the relation between (83) and (84). But there a certain form for G was employed whereas here (120) is a consequence of (117) and the general conditions imposed on G.

It will be noted that (122) differs from (115) only in the sign of the right hand side, but this difference is sufficient to prevent the application of the method, used for (115), to (122). It is still true that $\max|G_o|$ occurs at $v = 0$ because if G_o contains a logarithmic term it is infinite there and if it is bounded the same method as for (115) may be employed. However (116) takes the form $\gamma(v) - \gamma(\xi) = o(1)$ and we cannot deduce that $\max|G_o| = 0$.

If we multiply (117) by $G_o^*(v)e^{2i\alpha v}\, v(1-v^2)^{-1/2}$, the star indicating a complex conjugate, and integrate with respect to v from 0 to 1 we obtain

$$\int_0^1 \frac{v}{(1-v^2)^{1/2}} |G_o(v)|^2\, e^{2i\alpha v}\, dv = -\frac{1}{\pi}\int_0^1\int_0^1 \left\{\frac{vt}{(1-v)(1-t)}\right\}^{\frac{1}{2}} \frac{G_o(t)G_o{}^*(v)e^{-i\alpha t+i\alpha v}}{t+v}\, dt\, dv$$

$$= -\frac{1}{\pi}\int_0^\infty \left|\int_0^1 \left(\frac{t}{1-t}\right)^{1/2} G_o(t)e^{-i\alpha t-xt}dt\right|^2 dx \,.$$

Since the right hand side is real

$$(123) \qquad \int_0^1 \frac{v}{(1-v^2)^{1/2}}\, |G_o(v)|^2 \sin 2\alpha v\, dv = 0.$$

If $|\alpha| \leq \frac{1}{2}\pi$ this is possible only if $G_o(v) \equiv 0$ so that uniqueness is proved for $|\alpha| \leq \frac{1}{2}\pi$.

An extension of this range can be obtained by observing that

$$\mathcal{I}\,\frac{G_o^{*}(v)e^{i\alpha v}}{(1-v)^{1/2}v^{1/2}}\int_0^1 \frac{G_o(t)e^{-i\alpha t}dt}{(1-t)^{1/2}t^{1/2}(t+v)}$$

is integrable at the origin on account of (119). Hence if (121) be multiplied by $G_o^{*}(v)\, e^{2i\alpha v}/v(1-v^2)^{\frac{1}{2}}$ and the imaginary part be integrated from 0 to 1 we obtain

$$(124)\quad \int_0^1 \frac{|G_o(v)|^2}{(1-v^2)^{1/2}} \;\frac{\sin 2\alpha v}{v}\; dv = 0.$$

A combination of (123) and (124) is

$$\int_0^1 \frac{|G_o(v)|^2}{(1-v^2)^{1/2}}\;(4\pi^2 v - \frac{\alpha^2}{v}) \sin 2\alpha v dv = 0$$

which implies $G_o(v) \equiv 0$ for $|\alpha| \leq \pi$.

APPENDIX A

In this appendix we derive the formula (3) for

$$\int_0^{2\pi} \frac{\exp\left[-i\alpha\{r^2+r_1^2 - 2rr_1\cos(\varphi-\varphi_1)\}^{1/2}\right]}{\{r^2+r_1^2 -2rr_1 \cos(\varphi-\varphi_1)\}^{1/2}}\, d\varphi_1 .$$

Suppose $r > r_1$ and write $b = r_1/r$ so that $b < 1$. Then one integral we have to calculate is

$$\int_0^{2\pi} \frac{\cos\left[\gamma\{1+b^2-2b \cos(\varphi-\varphi_1)\}^{1/2}\right]}{r\{1+b^2-2b \cos(\varphi-\varphi_1)\}^{1/2}}\, d\varphi_1$$

where $\gamma = \alpha r$. Put $\varphi_1-\varphi = \theta$ and then in the interval $(-\varphi,0)$ replace θ by $2\pi+\theta$. The result is

$$\int_0^{2\pi} \frac{\cos\{\gamma(1+b^2-2b\cos\theta)^{1/2}\}}{r(1+b^2-2b\cos\theta)^{1/2}}\, d\theta$$

$$= 4\int_0^{\frac{1}{2}\pi} \frac{\cos\{\gamma(1+b^2-2b\cos 2\chi)^{1/2}\}}{r(1+b^2-2b\cos 2\chi)}\, d\chi$$

by changing θ to $2\pi-\theta$ in the interval $(\pi,2\pi)$ and then putting $\theta=2\chi$. The substituion $\dfrac{\sin 2\chi}{\cos 2\chi - b} = \tan\theta$ now gives

$$2\int_0^{\pi} \frac{\cos\left[\gamma\{(1-b^2\sin^2\theta)^{1/2}-b\cos\theta\}\right]}{r(1-b^2\sin^2\theta)^{1/2}}\, d\theta$$

$$= 4\int_0^{\frac{1}{2}\pi} \frac{\cos\{\gamma(1-b^2\sin^2\theta)^{1/2}\}\cos(\gamma\, b\cos\theta)}{r(1-b^2\sin^2\theta)^{1/2}}\, d\theta$$

$$= r\int_0^{r_1} \frac{\cos\{\alpha(r^2-x^2)^{1/2}\}\cos\{\alpha(r_1^2-x^2)^{1/2}\}}{(r^2-x^2)^{1/2}(r_1^2-x^2)^{1/2}}\, dx.$$

When $r < r_1$ just interchange r and r_1 in this last formula because of the invariance of the original integral under such an interchange. Hence

$$\int_0^{2\pi} \frac{\cos\left[\alpha\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}\right]}{\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}}\,d\varphi_1$$

$$= 4\int_0^{\min(r,r_1)} \frac{\cos\{\alpha(r^2-x^2)^{1/2}\}\cos\{\alpha(r_1^2-x^2)^{1/2}\}}{(r^2-x^2)^{1/2}(r_1^2-x^2)^{1/2}}\,dx.$$

The integral involving the sine can be dealt with similarly but for our purposes it is more convenient to proceed otherwise. We have

$$\int_0^{2\pi} \frac{\sin\left[\alpha\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}\right]}{\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}}\,d\varphi_1$$

$$= \left(\tfrac{1}{2}\alpha\pi\right)^{1/2}\int_0^{2\pi} \frac{J_{1/2}\left[\alpha\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}\right]}{\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/4}}\,d\varphi_1$$

$$= \alpha\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \sin\theta J_0\left[\alpha\{r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)\}^{1/2}\sin\theta\right]d\theta d\varphi_1$$

after using Sonine's first finite integral for Bessel functions (Watson [30]). Use of the addition theorem for Bessel functions now gives

$$2\pi\alpha\int_0^{\frac{1}{2}\pi} \sin\theta\, J_0(\alpha\, r\sin\theta)J_0(\alpha\, r_1\sin\theta)d\theta.$$

Now it has been proved elsewhere (Jones [4]) that

$$\int_0^{\frac{1}{2}\pi} J_{\nu-\frac{1}{2}}(z \sin \theta)\, J_{\mu+\frac{1}{2}}(Z \sin \theta) \sin^{\nu-\mu} \theta \cos^{2\lambda} \theta d\theta$$

$$= z^{\nu-\frac{1}{2}} Z^{\lambda+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} J_{\mu-\lambda}(Z \sin \theta) \frac{J_{\nu+\lambda}\{(z^2-Z^2\cos^2\theta)^{1/2}\}}{(z^2-Z^2\cos^2\theta)^{\frac{1}{2}(\nu+\lambda)}}$$

$$\cdot \sin^{\mu-\lambda+1} \theta \cos^{2\lambda} \theta d\theta$$

where $\mathcal{R}(\lambda) > -\frac{1}{2}$, $\mathcal{R}(\mu) > \mathcal{R}(\lambda) - 1$, $\mathcal{R}(\nu) > -\frac{1}{2}$. Using this result with $\nu = \frac{1}{2}$, $\mu = -\frac{1}{2}$, $\lambda = 0$, $z = \alpha r_1$ and $Z = \alpha r$, we obtain

$$4 \int_0^{\frac{1}{2}\pi} \frac{\cos(\alpha r \sin \theta) \sin \{\alpha(r_1^2 - r^2 \cos^2 \theta)^{1/2}\}}{(r_1^2 - r^2 \cos^2 \theta)^{1/4}} d\theta.$$

Hence

$$\int_0^{2\pi} \frac{\sin[\alpha\{r^2+r_1^2-2rr_1 \cos(\varphi-\varphi_1)\}^{1/2}]}{\{r^2+r_1^2-2rr_1 \cos(\varphi-\varphi_1)\}^{1/2}} d\varphi_1$$

$$= 4 \int_0^{r} \frac{\cos \alpha(r^2-x^2)^{1/2} \sin \alpha(r_1^2-x^2)^{1/2}}{(r^2-x^2)^{1/2}(r_1^2-x^2)^{1/2}} dx.$$

Since r and r_1 can be interchanged on the left hand side without affecting the result the same must be true of the right hand side. This can be confirmed in the above procedure by taking $z = \alpha r$, $Z = \alpha r_1$. An independent verification is to note that

$$\int_{-\infty}^{\infty} \frac{\exp \{(z^2-a^2)^{1/2}-(z^2-b^2)^{1/2}\}}{(z^2-a^2)^{1/2}(z^2-b^2)^{1/2}} dz = 0.$$

Here a and b are real with $0 < a < b$ and the branch lines are $(-\infty,-b)$, $(-\infty,-a)$, (a,∞) and (b,∞) with $(z^2-a^2)^{1/2} = ia$ at $z = 0$. The contour of integration lies just above the real axis and the equation follows by deforming the contour to infinity. If we take the imaginary part of this equation we find

$$\int_0^b \frac{\sin(a^2-x^2)^{1/2}\cos(b^2-x^2)^{1/2}}{(a^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\,dx =$$

$$\int_0^a \frac{\cos(a^2-x^2)^{1/2}\sin(b^2-x^2)^{1/2}}{(a^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\,dx$$

and the restriction $a < b$ can obviously be dropped now.

APPENDIX B

In this appendix we consider the evaluation of

$$\int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}e^{-i\alpha(t-z)^{1/2}}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz$$

where x and t are non-negative real but, for the purposes of this calculation, α can be an arbitrary complex number. Write the integral as

$$\frac{1}{2}\int_0^x \frac{\exp\{i\alpha(x-z)^{1/2}-i\alpha(t-z)^{1/2}\}+\exp\{-i\alpha(x-z)^{1/2}-i\alpha(t-z)^{1/2}\}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz.$$

In the first integrand make the change of variable $(t-z)^{1/2}-(x-z)^{1/2} = y$ and in the second integrand put $(t-z)^{1/2}+(x-z)^{1/2} = y$. Thus we obtain

$$\int_{t^{1/2}-x^{1/2}}^{(t-x)^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy - \int_{t^{1/2}+x^{1/2}}^{(t-x)^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy.$$

If $t > x$ the integration in both is along the real axis and does not include the origin. If $t < x$ the upper limit of integration is positive imaginary (with the convention on square roots adopted in the main text) and the lower limits are negative and positive real respectively. In this case the contour of integration can be deformed into the real axis with an indentation round the simple pole at the origin. Hence

$$\int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}e^{-i\alpha(t-z)^{1/2}}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz = \int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{e^{-i\alpha y}}{y}\,dy - \pi i H(x-t) \tag{B1}$$

the principal value of the integral being understood and H(x) being the Heaviside unit function.

Since (B1) is valid for arbitrary complex α we can replace α by $-\alpha$ and add the resulting equation to (B1). In this way we obtain

$$\text{(B2)}\quad \int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}\cos\{\alpha(t-z)^{1/2}\}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz = \int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{\cos \alpha y}{y}\,dy - \pi i H(x-t).$$

Similarly

$$\text{(B3)}\quad \int_0^x \frac{\cos\{\alpha(x-z)^{1/2}\}\sin\{\alpha(t-z)^{1/2}\}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz = \int_{t^{1/2}-x^{1/2}}^{t^{1/2}+x^{1/2}} \frac{\sin \alpha y}{y}\,dy.$$

Interchanging x and t in (B2) gives

$$\text{(B4)}\quad \int_0^t \frac{\cos\{\alpha(t-z)^{1/2}\}\cos\{\alpha(x-z)^{1/2}\}}{(t-z)^{1/2}(x-z)^{1/2}}\,dz = \int_{x^{1/2}-t^{1/2}}^{x^{1/2}+t^{1/2}} \frac{\cos \alpha y}{y}\,dy - \pi i H(t-x).$$

The subtraction of (B4) from (B2) leads to

$$\text{(B5)}\quad \int_t^x \frac{\cos\{\alpha(x-z)^{1/2}\}\cos\{\alpha(t-z)^{1/2}\}}{(x-z)^{1/2}(t-z)^{1/2}}\,dz = -\pi i \operatorname{sgn}(x-t).$$

since

$$\int_{t^{1/2}-x^{1/2}}^{x^{1/2}-t^{1/2}} \frac{\cos \alpha y}{y}\,dy = 0.$$

APPENDIX C

This appendix is concerned with the solution of (11) at low frequencies when α is small. The method of approximation is that of truncation of the series expansion for the kernel. The error so introduced can be estimated but will not be considered further here. To illustrate the process we neglect the powers above α^2 in the kernel. The resulting integral equation is

$$\int_0^1 wf(w)\left\{\frac{2w}{w^2-v^2} - 2i\alpha - \alpha^2 w\right\} dw = F^{(0)}(v)$$

or

$$\text{(C1)} \qquad \int_0^1 \frac{2w^2 f(w)}{w^2-v^2}\, dw = F^{(0)}(v) + a_0\alpha + a_1\alpha^2$$

where the constants a_0 and a_1 are defined by $a_0 = 2i\int_0^1 wf(w)dw$, $a_1 = \int_0^1 w^2 f(w)dw$. Making the change of variables $w^2 = x$, $v^2 = y$ in (C1) we have

$$\int_0^1 \frac{x^{1/2}\, f(x^{1/2})}{x-y}\, dx = F^{(0)}(y^{1/2}) + a_0\alpha + a_1\alpha^2$$

which is of the same type as (13) with solution

$$x^{1/2} f(x^{1/2}) = -\frac{1}{\pi^2 x^{1/2}(1-x)^{1/2}} \int_0^1 \frac{y^{1/2}(1-y)^{1/2}}{y-x} \{F^{(0)}(y^{1/2})$$

$$+ a_0\alpha + a_1\alpha^2\}dy + \frac{c}{x^{1/2}(1-x)^{1/2}} = -\frac{x^{1/2}}{\pi^2(1-x)^{1/2}} \int_0^1 \frac{(1-y)^{1/2}}{y^{1/2}(y-x)} \times$$

$$\cdot\{F^{(0)}(y^{1/2}) + a_0\alpha + a_1\alpha^2\}\, dy$$

on choosing c so as to eliminate the singularity at x = 0. Since

$$\int_0^1 \frac{(1-y)^{1/2}}{y^{1/2}(y-x)}\,dy = -\pi,$$

$$f(x^{1/2}) = \frac{a_0\alpha + a_1\alpha^2}{\pi(1-x)^{1/2}} - \frac{1}{\pi^2(1-x)^{1/2}} \int_0^1 \frac{(1-y)^{1/2}}{y^{1/2}(y-x)} F^{(0)}(y^{1/2})\,dy.$$

Thus

$$\text{(C2)}\quad f(w) = \frac{a_0\alpha + a_1\alpha^2}{\pi(1-w^2)^{1/2}} - \frac{2}{\pi^2(1-w^2)^{1/2}} \int_0^1 \frac{(1-v^2)^{1/2}}{v^2-w^2} F^{(0)}(v)\,dv.$$

The constants a_0 and a_1 can be determined by multiplying by w and w^2 and integrating over (0,1) in the two cases.

For instance, with an incident plane wave, we have from (19)

$$F^{(0)}(v) = 4 \cos \alpha\, v + 4\, \alpha\, v\, \mathrm{Si}(\alpha v).$$

In principle, there is no necessity to approximate $F^{(0)}$ in (C2) but, for simplicity, we shall use only the first two terms in its expansion in powers of α i.e.

$$F^{(0)}(v) \approx 4 + 2\alpha^2 v^2.$$

Since

$$\int_0^1 \frac{(1-v^2)^{1/2}}{v^2-w^2}\,dv = -\frac{1}{2}\pi$$

and

$$\int_0^1 \frac{(1-v^2)^{1/2}\, v^2}{v^2-w^2}\,dv = \frac{1}{2}\pi\left(\frac{1}{2} - w^2\right)$$

we have

(C3) $$f(w) = \frac{a_0\alpha + a_1\alpha^2}{\pi(1-w^2)^{1/2}} + \frac{4}{\pi(1-w^2)^{1/2}} \{1 - \frac{1}{2}\alpha^2(\frac{1}{2} - w^2)\}.$$

Hence

$$a_0 = 2i\int_0^1 wf(w)dw = \frac{2i}{\pi}(a_0\alpha + a_1\alpha^2) + \frac{8i}{\pi}(1 + \frac{1}{12}\alpha^2)$$

and

$$a_1 = \int_0^1 w^2f(w)dw = \frac{1}{4}(a_0\alpha + a_1\alpha^2) + 1 + \frac{1}{8}\alpha^2.$$

The solution of these simultaneous equations is

$$a_0 = \frac{8i}{\pi}\{1 + \frac{2i\alpha}{\pi}\}$$

$$a_1 = 1$$

correct to the terms in α and α^0 respectively. Substitution in (C3) gives

$$f(w) = \frac{4}{\pi}\{1 + \frac{2i\alpha}{\pi} + \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\alpha^2\}(1-w^2)^{-1/2} - \frac{2\alpha^2}{\pi}(1-w^2)^{1/2} + O(\alpha^3).$$

This is in agreement with the results of Bazer and Brown [7].

The calculation of f correct to $O(\alpha^2)$ involves the determination of a_0 and a_1 the first two moments of f. To obtain the terms which are $O(\alpha^3)$ it would be necessary to solve three equations for the first three moments. Clearly, although all the integrals required are relatively simple, this approach does not provide the expansion in powers of α as rapidly as the integral equation found earlier. (Jones [4]).

Finally note that the determination of a_0 is virtually equivalent to finding the scattering coefficient.

APPENDIX D

In this Appendix the solution of the integral equation

$$\text{(D1)} \qquad \int_0^\infty f(w) \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} dw = h(v) \qquad (v > 0)$$

is derived. We find the solution such that f is integrable. The method of solution is not difficult once certain integrals containing Bessel functions are available. These we shall now obtain.

Let

$$\text{(D2)} \qquad M(x,y) = \frac{4xy\, H_1^{(2)}(\alpha x)\, J_0(\alpha y) - 4y^2\, H_0^{(2)}(\alpha x)\, J_1(\alpha y)}{x^2-y^2}$$

where the usual notation of Bessel and Hankel functions is employed. Consider

$$\frac{1}{4} \int_C M(z,y)\, \frac{e^{i\alpha(z+w)}}{z+w}\, dz$$ where $y > 0$, $w > 0$ and C is the contour shown in Figure 1. The only singularity inside the contour is the simple pole $z = y$. Hence

$$\frac{1}{4} \int_C M(z,y)\, \frac{e^{i\alpha(z+w)}}{z+w}\, dz = -\frac{2}{\alpha}\, \frac{e^{i\alpha(y+w)}}{y+w}$$

after using the Wronskian relation

$$H_1^{(2)}(\alpha y)\, J_0(\alpha y) - H_0^{(2)}(\alpha y)\, J_1(\alpha y) = 2i/\pi\alpha y.$$

On the large circle of radius r the asymptotic formula

$$H_\nu^{(2)}(\alpha z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(\alpha z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}$$

(valid for $-2\pi < \arg z < \pi$) may be employed and then we see that the contribution of the large circle tends to zero as $r \to \infty$. On the straight

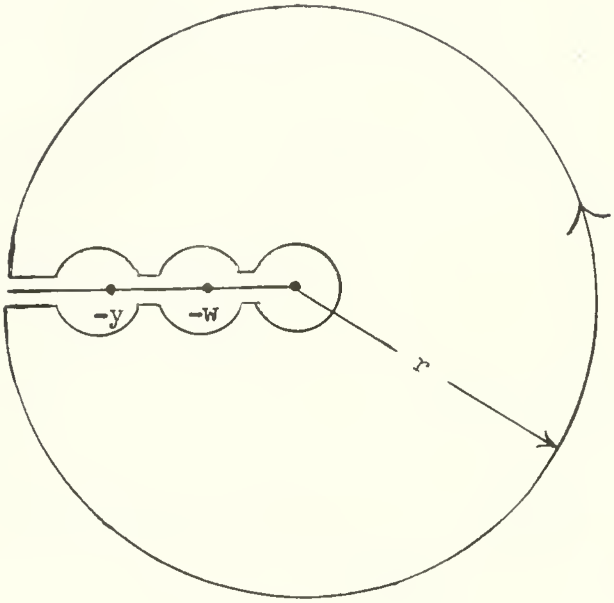

Figure 1

portions of the negative real axis we put $z = x\, e^{\pm \pi i}$ and use the fact

$$H_{\nu}^{(2)}(xe^{\pi i}) - H_{\nu}^{(2)}(xe^{-\pi i}) = -2i \sin \nu\pi\, H_{\nu}^{(2)}(x) + 4e^{\nu\pi i}\, J_{\nu}(x).$$

For the indentations about $z = -y$ and $z = -w$ we need

$$H_{\nu}^{(2)}(xe^{\pi i}) + H_{\nu}^{(2)}(xe^{-\pi i}) = 2 \cos \nu\pi\, H_{\nu}^{(2)}(x).$$

There is no contribution from the indentation around the origin as its radius shrinks to zero. The net result is

$$\text{(D3)} \qquad \int_0^\infty K(x,y)\, \frac{e^{-i\alpha(x-w)}}{x-w}\, dx = \frac{2}{\alpha} \left\{ \frac{e^{i\alpha(y+w)}}{y+w} + \frac{e^{-i\alpha(y-w)}}{y-w} \right\} - \frac{1}{2}\pi i M(w,y)$$

where

$$\text{(D4)} \qquad K(x,y) = \frac{4xy\, J_1(\alpha x)\, J_0(\alpha y) - 4y^2 J_0(\alpha x) J_1(\alpha y)}{x^2-y^2}\,.$$

Similarly, it can be shown that

$$\text{(D5)} \qquad \int_0^\infty K(x,y)\, \frac{e^{-i\alpha(x+w)}}{x+w}\, dx = \frac{2}{\alpha} \left\{ \frac{e^{i\alpha(y-w)}}{y-w} + \frac{e^{-i\alpha(y+w)}}{y+w} \right\} - \frac{1}{2}\pi i M(w,y).$$

By replacing $H_{\nu}^{(2)}$ and $e^{i\alpha z}$ by $H_{\nu}^{(1)}$ and $e^{-i\alpha z}$ we obtain in a similar manner

$$\text{(D6)} \qquad \int_0^\infty K(x,y)\, \frac{e^{i\alpha(x-w)}}{x-w}\, dx = \frac{2}{\alpha} \left\{ \frac{e^{-i\alpha(y+w)}}{y+w} + \frac{e^{i\alpha(y-w)}}{y-w} \right\} + \frac{1}{2}\pi i M_1(w,y),$$

$$\text{(D7)} \qquad \int_0^\infty K(x,y)\, \frac{e^{i\alpha(x+w)}}{x+w}\, dx = \frac{2}{\alpha} \left\{ \frac{e^{-i\alpha(y-w)}}{y-w} + \frac{e^{i\alpha(y+w)}}{y+w} \right\} + \frac{1}{2}\pi i M_1(w,y)$$

where M_1 is the same as M with $H_\nu^{(1)}$ taking the place of $H_\nu^{(2)}$.

Adding (D3), (D5), (D6) and (D7) we find

$$\text{(D8)} \qquad \int_0^\infty K(x,y) \left\{ \frac{\cos \alpha(x+w)}{x+w} + \frac{\cos \alpha(x-w)}{x-w} \right\} dx = \frac{4}{\alpha} \left\{ \frac{\cos \alpha(y+w)}{y+w} + \frac{\cos \alpha(y-w)}{y-w} \right\} - \pi L(w,y)$$

where

$$\text{(D9)} \qquad L(x,y) = \frac{4xy\, Y_1(\alpha x) J_0(\alpha y) - 4y^2 Y_0(\alpha x) J_1(\alpha y)}{x^2-y^2}$$

so that

$$\text{(D10)} \qquad M(x,y) = K(x,y) - iL(x,y)$$

and

$$M_1(x,y) = K(x,y) + iL(x,y).$$

Next, it follows from (D8) that

$$\text{(D11)} \qquad \int_0^\infty L(v,y) \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} dv = \frac{1}{\pi} \int_0^\infty \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} \left[\frac{4}{\alpha} \left\{ \frac{\cos \alpha(y+v)}{y+v} + \frac{\cos \alpha(y-v)}{y-v} \right\} - \int_0^\infty K(x,y) \left\{ \frac{\cos \alpha(x+v)}{x+v} + \frac{\cos \alpha(x-v)}{x-v} \right\} dx \right] dv.$$

Now

$$\int_0^\infty \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} \left\{ \frac{\cos \alpha(y+v)}{y+v} + \frac{\cos \alpha(y-v)}{y-v} \right\} dv$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\}\left\{ \frac{\cos\alpha(y+v)}{y+v} + \frac{\cos\alpha(y-v)}{y-v} \right\} dv$$

and the integral on the right hand side can be evaluated by expressing the cosines as exponentials and using formulae such as

$$\int_{-\infty}^{\infty} \frac{e^{2i\alpha v}\,dv}{(v-w)(v-y)} = \pi^2\delta(w-y)\, e^{2i\alpha y} + \pi i(e^{2i\alpha w} - e^{2i\alpha y})/(w-y).$$

Thus

$$\text{(D12)}\qquad \begin{aligned} &\int_{0}^{\infty} \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\}\left\{ \frac{\cos\alpha(y+v)}{y+v} + \frac{\cos\alpha(y-v)}{y-v} \right\} dv \\ &= \pi^2\delta(w-y) + \pi\left\{ \frac{\sin\alpha(w+y)}{w+y} - \frac{\sin\alpha(y-w)}{y-w} \right\} \end{aligned}$$

for the values of y and w involved. Inserting this result in (D11) we obtain

$$\text{(D13)}\qquad \begin{aligned} &\int_{0}^{\infty} L(v,y)\left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} dv = \frac{4\pi}{\alpha}\,\delta(w-y) + \frac{4}{\alpha}\left\{ \frac{\sin\alpha(w+y)}{w+y} \right. \\ &\left. - \frac{\sin\alpha(y-w)}{y-w} \right\} - \pi K(w,y) - \int_{0}^{\infty} K(x,y)\left\{ \frac{\sin\alpha(x+w)}{x+w} - \frac{\sin\alpha(x-w)}{x-w} \right\} dx \\ &= \frac{4\pi}{\alpha}\,\delta(w-y) - \pi K(w,y) \end{aligned}$$

from (D3), (D5), (D6) and (D7). Since, from (D5) and (D6),

$$\int_{0}^{\infty} K(v,y)\left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} dv = -\pi i K(w,y)$$

we see, recalling (D10), that

$$\text{(D14)}\qquad \int_0^\infty M(v,y)\left\{\frac{e^{-i\alpha(w+v)}}{w+v}+\frac{e^{-i\alpha(w-v)}}{w-v}\right\}dv = -\frac{4\pi i}{\alpha}\,\delta(w-y).$$

Multiply (D1) by $i\alpha M(v,y)/4\pi$ and integrate with respect to v from 0 to ∞. On account of (D14)

$$\int_0^\infty f(w)\delta(w-y)dw = \frac{i\alpha}{4\pi}\int_0^\infty h(v)M(v,y)dv.$$

Hence the solution of the integral equation is

$$\text{(D15)}\qquad f(y) = \frac{i\alpha}{4\pi}\int_0^\infty h(v)M(v,y)dv.$$

When $\alpha \to 0$, $\alpha M(x,y) \to 8iy/\pi(x^2-y^2)$ and in the limit (D15) gives

$$\text{(D16)}\qquad f(y) = -\frac{2y}{\pi^2}\int_0^\infty \frac{h(v)dv}{v^2-y^2}.$$

That this is indeed the solution with integrable f and $\alpha = 0$ can be confirmed by an independent analysis. The integral equation is now

$$\int_0^\infty \frac{2wf(w)}{w^2-v^2}\,dw = h(v).$$

As in Appendix C the general solution of this is given by

$$f(w^{1/2}) = -\frac{1}{\pi^2 w^{1/2}}\int_0^\infty \frac{v^{1/2}h(v^{1/2})}{v-w}\,dv + \frac{C}{w^{1/2}}$$

or

$$f(w) = -\frac{2}{\pi^2 w}\int_0^\infty \frac{u^2h(u)}{u^2-w^2}\,du + \frac{C}{w}.$$

On choosing C so as to eliminate the non-integrable singularity at the origin we recover (D16). The presence of the constant C in the general solution here indicates that in suitable circumstances there may be solutions of (D1) in which f is not integrable but the left hand side has a meaning in spite of that.

For the related integral equation

$$\text{(D17)} \qquad \int_0^\infty f(w)\left\{\frac{e^{-i\alpha(w-v)}}{w-v} - \frac{e^{-i\alpha(w+v)}}{w+v}\right\} dw = h(v) \qquad (v > 0)$$

start from

$$\text{(D18)} \qquad M_2(x,y) = \frac{4xyH_0^{(2)}(\alpha x)J_1(\alpha y) - 4x^2H_1^{(2)}(\alpha x)J_0(\alpha y)}{x^2-y^2} .$$

By the same method as the preceding we show that

$$\text{(D19)} \qquad \int_0^\infty K_1(x,y)\,\frac{e^{-i\alpha(x-w)}}{x-w}\,dx = \frac{2}{\alpha}\left\{\frac{e^{i\alpha(y+w)}}{y+w} - \frac{e^{-i\alpha(y-w)}}{y-w}\right\} - \tfrac{1}{2}\pi i M_2(w,y),$$

$$\text{(D20)} \qquad \int_0^\infty K_1(x,y)\,\frac{e^{-i\alpha(x+w)}}{x+w}\,dx = \frac{2}{\alpha}\left\{\frac{e^{i\alpha(y-w)}}{y-w} - \frac{e^{-i\alpha(y+w)}}{y+w}\right\} + \frac{1}{2}\pi i M_2(w,y),$$

$$\text{(D21)} \qquad \int_0^\infty K_1(x,y)\,\frac{e^{i\alpha(x-w)}}{x-w}\,dx = \frac{2}{\alpha}\left\{\frac{e^{-i\alpha(y+w)}}{y+w} - \frac{e^{i\alpha(y-w)}}{y-w}\right\} + \frac{1}{2}\pi i M_3(w,y),$$

$$\text{(D22)} \qquad \int_0^\infty K_1(x,y)\,\frac{e^{i\alpha(x+w)}}{x+w}\,dx = \frac{2}{\alpha}\left\{\frac{e^{-i\alpha(y-w)}}{y-w} - \frac{e^{i\alpha(y+w)}}{y+w}\right\} - \frac{1}{2}\pi i M_3(w,y)$$

where

$$\text{(D23)} \qquad K_1(x,y) = \frac{4xyJ_0(\alpha x)J_1(\alpha y) - 4x^2J_1(\alpha x)J_0(\alpha y)}{x^2-y^2}$$

and M_3 is the same as M_2 with $H_\nu^{(2)}$ replaced by $H_\nu^{(1)}$. It follows from (D19)-(D22) that

$$\text{(D24)} \quad \int_0^\infty K_1(x,y)\left\{\frac{\cos\alpha(x+w)}{x+w} - \frac{\cos\alpha(x-w)}{x-w}\right\} dx = \frac{4}{\alpha}\left\{\frac{\cos\alpha(y-w)}{y-w}\right.$$

$$\left. - \frac{\cos\alpha(y+w)}{y+w}\right\} + \pi L_1(w,y)$$

where $K_1(x,y) - iL_1(x,y) = M_2(x,y)$.

Since

$$\int_0^\infty \left\{\frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\}\left\{\frac{\cos\alpha(y+v)}{y+v} - \frac{\cos\alpha(y-v)}{y-v}\right\} dv$$

$$= \pi^2\delta(w-y) - \pi\left\{\frac{\sin\alpha(w-y)}{w-y} + \frac{\sin\alpha(w+y)}{w+y}\right\}$$

for $w > 0$, $y > 0$ it follows from (D24) that

$$\int_0^\infty L_1(v,y)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dv = \frac{4\pi}{\alpha}\,\delta(w-y) - \frac{4}{\alpha}\left\{\frac{\sin\alpha(w-y)}{w-y} + \right.$$

$$\left. + \frac{\sin\alpha(w+y)}{w+y}\right\} + \pi K_1(w,y) - \int_0^\infty K_1(x,y)\left\{\frac{\sin\alpha(w-x)}{w-x} + \frac{\sin\alpha(w+x)}{w+x}\right\} dx$$

$$= \frac{4\pi}{\alpha}\,\delta(w-y) + \pi K_1(w,y)$$

from (D19)-(D22). Also

$$\int_0^\infty K_1(v,y)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dv = \pi i K_1(w,y)$$

so that

$$\int_0^\infty M_2(v,y)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dv = -\frac{4\pi i}{\alpha}\,\delta(w-y).$$

Hence, if (D17) be multiplied by $M_2(v,y)$ and integrated from 0 to ∞, we obtain

$$\int_0^\infty f(w)\,\frac{4\pi i}{\alpha}\,\delta(w-y)dw = \int_0^\infty h(v)M_2(v,y)dv.$$

Consequently the solution of (D17) is

$$\text{(D24)} \qquad f(y) = \frac{\alpha}{4\pi i}\int_0^\infty h(v)\,M_2(v,y)dv.$$

It is of interest to note that in the course of the preceding analysis it has been demonstrated that the equation

$$f(v) = \lambda\int_0^\infty f(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw$$

has an eigenvalue $\lambda = i/\pi$ and a corresponding eigenfunction $f(v) = K(v,y)$. The integral equation

$$f(v) = \lambda\int_0^\infty f(w)\left\{\frac{e^{-i\alpha(w-v)}}{w-v} - \frac{e^{-i\alpha(w+v)}}{w+v}\right\} dw$$

has an eigenvalue $\lambda = i/\pi$ with associated eigenfunction $f(v) = K_1(v,y)$.

APPENDIX E

In this Appendix an alternative evaluation of

$$\int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} \qquad (0 < w < 1)$$

is given. A Fourier transform of the integral is

$$\int_{-\infty}^{\infty} e^{-i\beta t} \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)}\, dt$$

$$= \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{\pi i\ e^{-i\beta v}\ \mathrm{sgn}\ \beta}{v-w}\, dv.$$

Take branch lines from $-\infty$ to 0 and from 1 to ∞. Then we specify the branches of $v^{1/2}$ and $(1-v)^{1/2}$ by saying that $|\arg v| < \pi$ and that $(1-v)^{1/2} = (v-1)^{1/2}\, e^{(-1/2)\pi i}$ with $2\pi > \arg(v-1) > 0$. Now, for $\beta > 0$, extend the interval of integration to $(-\infty,\infty)$, the path lying entirely below the branch lines. Then the right hand side is

$$\int_{-\infty}^{\infty} \left(\frac{1-v}{v}\right)^{1/2} \frac{\pi i\ e^{-i\beta v}}{v-w}\, dv + \int_1^{\infty} \left(\frac{v-1}{v}\right)^{1/2} \frac{\pi\ e^{-i\beta v}}{v-w}\, dv - \int_0^{\infty} \left(\frac{1+v}{v}\right)^{1/2} \frac{\pi\ e^{i\beta v}}{v+w}\, dv.$$

When the path of the first integral is completed by an indentation below $v = w$ the contour can be deformed to infinity to give no contribution so that the value of the first integral comes from the indentation. Consequently we obtain when $\beta > 0$

$$\text{(E1)} \quad \pi^2 \left(\frac{1-w}{w}\right)^{1/2} e^{-i\beta w} + \pi \int_1^{\infty} \left(\frac{v-1}{v}\right)^{1/2} \frac{e^{-i\beta v}}{v-w}\, dv - \pi \int_0^{\infty} \left(\frac{1+v}{v}\right)^{1/2} \frac{e^{i\beta v}}{v+w}\, dv.$$

When $\beta < 0$ we extend the interval to $(-\infty,\infty)$ but with the path lying above the branch lines. On repeating the process and remembering the presence of the factor sgn β we find (E1) also when $\beta < 0$.

An inverse Fourier transform now supplies the original integral by

$$\int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\beta t} \left\{ \pi^2 \left(\frac{1-w}{w}\right)^{1/2} e^{-i\beta w} \right.$$

$$\left. + \pi \int_1^{\infty} \left(\frac{v-1}{v}\right)^{1/2} \frac{e^{-i\beta v}}{v-w}\, dv - \pi \int_0^{\infty} \left(\frac{1+v}{v}\right)^{1/2} \frac{e^{i\beta v}}{v+w}\, dv \right\} d\beta .$$

Since $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\beta t}\, d\beta = \delta(t)$ we obtain, for $0 < w < 1$,

$$\text{(E2)} \qquad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \pi^2 \left(\frac{1-w}{w}\right)^{1/2} \delta(t-w) + \pi \left(\frac{t-1}{t}\right)^{1/2} \times$$

$$\frac{H(t-1)}{t-w} + \pi \left(\frac{1-t}{-t}\right)^{1/2} \frac{H(-t)}{t-w} .$$

In particular,

$$\text{(E3)} \qquad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \pi^2 \left(\frac{1-w}{w}\right)^{1/2} \delta(t-w) + \pi \left(\frac{t-1}{t}\right)^{1/2} \frac{H(t-1)}{t-w}$$

$$(0 < w < 1,\ t > 0),$$

$$\text{(E4)} \qquad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v+t)} = \frac{-\pi}{t+w} \left(\frac{1+t}{t}\right)^{1/2} \qquad (0 < w < 1,\ t > 0).$$

By multiplying (E4) by t and letting $t \to \infty$ we find

$$\text{(E5)} \qquad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{v-w} = -\pi \qquad (0 < w < 1).$$

Then, for $0 < w < 1$,

$$\text{(E6)} \quad \int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{(v-w)(v-t)}\, dv = \int_0^1 \left\{ \left(\frac{1-v}{v}\right)^{1/2} \frac{1}{v-w} + t \left(\frac{1-v}{v}\right)^{1/2} \frac{1}{(v-w)(v-t)} \right\} dt$$

$$= -\pi + \pi^2 w^{1/2}(1-w)^{1/2}\delta(t-w) + \pi\, t^{1/2}(t-1)^{1/2}\, \frac{H(t-1)}{t-w}$$

$$- \pi(-t)^{1/2}(1-t)^{1/2}\, \frac{H(-t)}{t-w}$$

from (E2) and (E5). In particular

$$\text{(E7)} \quad \int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{(v-w)(v-t)}\, dv = -\pi + \pi^2 w^{1/2}(1-w)^{1/2}\delta(t-w) + \frac{\pi t^{1/2}(t-1)^{1/2}}{t-w} H(t-1)$$

$$(0 < w < 1,\ t > 0),$$

$$\text{(E8)} \quad \int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{(v-w)(v+t)}\, dv = -\pi + \frac{\pi t^{1/2}(1+t)^{1/2}}{t+w} \qquad (0 < w < 1,\ t > 0).$$

By interchanging w and t in (E2) we obtain a formula for the integral on the left hand side valid for all w and $0 < t < 1$. Thus the only range not covered is that in which neither w nor t lies in (0,1). Elementary computations show that for these w and t

$$\int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \frac{\pi}{w-t} \left\{ \left(\frac{w-1}{w}\right)^{1/2} - \left(\frac{t-1}{t}\right)^{1/2} \right\}.$$

Inspection of this and (E2) reveals this can be conveniently combined with (E2) to give, if $w \notin (0,1)$

$$\text{(E9)} \quad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \frac{\pi}{w-t}\left[\left(\frac{w-1}{w}\right)^{1/2} - \left(\frac{t-1}{t}\right)^{1/2}\{H(t-1)+H(-t)\}\right].$$

Thus, for $w > 0$ and $t > 0$

$$\text{(E10)} \quad \int_0^1 \left(\frac{1-v}{v}\right)^{1/2} \frac{dv}{(v+w)(v-t)} = \frac{\pi}{t+w}\left\{\left(\frac{t-1}{t}\right)^{1/2} H(t-1) - \left(\frac{1+w}{w}\right)^{1/2}\right\}.$$

It may also be shown, in a similar way to that used for (E6), that

$$\text{(E11)} \quad \int_0^1 \frac{(1-v)^{1/2} v^{1/2}}{(v+w)(v-t)}\, dv = \frac{\pi}{t+w}\left\{w^{1/2}(1+w)^{1/2} + t^{1/2}(t-1)^{1/2} H(t-1)\right\} - \pi$$

for $w > 0$ and $t > 0$.

By changing the variable v to $1-v$ we deduce that

$$\text{(E12)} \quad \int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{dv}{(v-w)(v-t)} = \pi^2\left(\frac{w}{1-w}\right)^{1/2}\delta(t-w) + \frac{\pi}{w-t}\left(\frac{t}{t-1}\right)^{1/2}\{H(t-1)+H(-t)\}$$

$$(0 < w < 1)$$

$$\text{(E13)} \quad = \frac{\pi}{t-w}\left[\left(\frac{w}{w-1}\right)^{1/2} - \left(\frac{t}{t-1}\right)^{1/2}\{H(t-1)+H(-t)\}\right]$$

$$(w \notin (0,1))$$

and, in particular, that

$$\text{(E14)} \quad \int_0^1 \left(\frac{v}{1-v}\right)^{1/2} \frac{dv}{(v+w)(v-t)} = \frac{\pi}{t+w}\left\{\left(\frac{w}{1+w}\right)^{1/2} - \left(\frac{t}{t-1}\right)^{1/2} H(t-1)\right\}$$

$$(w > 0,\ t > 0).$$

Bibliography

[1] Bouwkamp, C.J., *Theoretische en numerieke behandeling van de buiging door een ronde opening*; Dissertation, University of Groningen, 1941.

[2] Chako, N., *On certain integrals in diffraction theory* (to be published).

[3] Nomura, Y., and Katsura, S., *Diffraction of electromagnetic waves by circular plate and circular hole*; Jour. Phys. Soc. Japan, Vol. 10, 1954, pp. 285-304.

[4] Jones, D.S., *A new method for calculating scattering, with particular reference to the circular disc*; Comm. Pure Appl. Math., Vol. 9, 1956, pp. 713-746, and N.Y.U. Res. Rep. EM-87, 1955.

[5] Akhiezer, N.I. and Akhiezer, A.N., *K zadache o diffraktsii elektromagnitnykh voln u krugovogo otverstiia v ploskom ekrane*; Dokl. Akad. Nauk, S.S.S.R., Vol. 109, 1956, pp. 53-56.

[6] MacCamy, R.C., and Heins, A.E., *On the scattering of waves by a disk*; ZAMP, Vol. 11, 1960, pp. 249-264.

[7] Bazer, J., and Brown, A., *Diffraction of scalar waves by a circular aperture*; Trans. I.R.E. AP-7, Special Supplement, 1959, pp. 12-20, and N.Y.U. Res. Rep. EM-144, 1959,

[8] Noble, B., *Integral equation perturbation methods in low-frequency diffraction*; Proc. Symp. on Electromagnetic Theory, University of Wisconsin, 1961, pp. 323-360.

[9] Collins, W.D., *On the solution of some axisymmetric boundary value problems by means of integral equations, V. Some scalar diffraction problems for circular discs*; Quart. J. Mech. App. Math., Vol. 14, 1961, pp. 101-117.

[10] Bazer, J., and Hochstadt, H., *Diffraction of scalar waves by a circular aperture, II*; Comm. Pure App. Math., Vol. 15, 1962, pp. 1-33, and N.Y.U. Res. Rep. EM-170, 1961.

[11] Williams, W.E., *The reduction of boundary value problems to Fredholm integral equations of the second kind*; ZAMP, Vol. 12, 1962, pp. 133-151.

[12] Magnus, W., An infinite system of linear equations arising in diffraction theory; N.Y.U. Res. Rep. EM-80, 1955.

[13] Lebedev, N.N., and Skal'skaya, I.P., A new method for solving the problem of the diffraction of electromagnetic waves by a thin conducting dick; Sov. Phys. Tech. Phys., Vol. 4, 1960, pp. 627-637.

[14] Collins, W.D., Some scalar diffraction problems for a spherical cap; Arch. Rat. Mech. Anal., Vol. 10, 1962, pp. 249-266.

[15] Levine, H., Diffraction by a circular aperture at high frequencies; N.Y.U. Res. Rep. EM-84, 1955.

[16] Levine, H., and Wu, T.T., Diffraction by an aperture at high frequencies; Technical Report 71, Stanford University, 1957.

[17] Seshadri, S.R., and Wu, T.T., High-frequency diffraction of plane electromagnetic waves by a circular aperture in an infinite plane screen; Craft Lab. Report No. 16, Harvard University, 1958.

[18] Clemmow, P.C., Edge currents in diffraction theory; Trans. I.R.E. AP-4, 1956, pp. 282-287.

[19] Millar, R.F., The diffraction of an electromagnetic wave by a circular aperture; Proc. I.E.E. Pt. C, Monograph 196R, 1955.

[20] Millar, R.F., The diffraction of an electromagnetic wave by a large aperture; Proc. I.E.E. Pt. C, Monograph 213R, 1956.

[21] Keller, J.B., Diffraction by an aperture, I; J. App. Phys., Vol. 28, 1957, pp. 426-444

[22] Keller, J.B., Errata: Diffraction by an aperture; J. App. Phys., Vol. 29, 1958, p. 744.

[23] Keller, J.B., Lewis, R.M., and Seckler, B.D., Diffraction by an aperture, II; J. App. Phys., Vol. 28, 1957, pp. 570-579.

[24] Karp, S.N., and Keller, J.B., Multiple diffraction by an aperture in a hard screen; Optica Acta, Vol. 8, 1961, pp. 61-71.

[25] Keller, J.B., and Buchal, R.N., Boundary layer problems in diffraction theory; Comm. Pure App. Math., Vol. 13, 1960, pp. 85-96.

[26] Mikhlin, S.G., Integral Equations; Pergamon Press, London, 1957.

[27] Jones, D.S., Diffraction by an edge and by a corner; Quart. J. Mech. App. Math. Vol. 5, 1952, pp. 363-378.

[28] Gilbert, L.P., Recherches analytiques sur la diffraction de la lumière; Mém. couronnés de l'Acad. R. des Sci. de Bruxelles, Vol. 31, 1863, pp. 1-52.

[29] Jones, D.S., On the scattering cross section of an obstacle; Phil. Mag., Vol. 46, 1955, pp. 957-962.

[30] Watson, G.N., Theory of Bessel Functions; Cambridge University Press, 2nd ed., 1944.

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It is shown how to convert the integral equation for the pressure distribution on a circular disc in an axi-symmetric sound wave into a singular integral equation of the first kind. This singular integral equation has a simple kernel and can be transformed easily into an integral equation of the second kind which is especially useful at high frequencies. Although a direct iteration on this last integral equation fails an indirect method is devised which ensures that each iterate is of lower order than the preceding at high frequencies. The form of the general term in the iteration is given together with the first terms of its asymptotic behaviour. It is relatively simple to estimate the error caused by stopping at any particular iterate.

Detailed calculations are made for a plane wave striking the disc at normal incidence.

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